GRAPH THEORY

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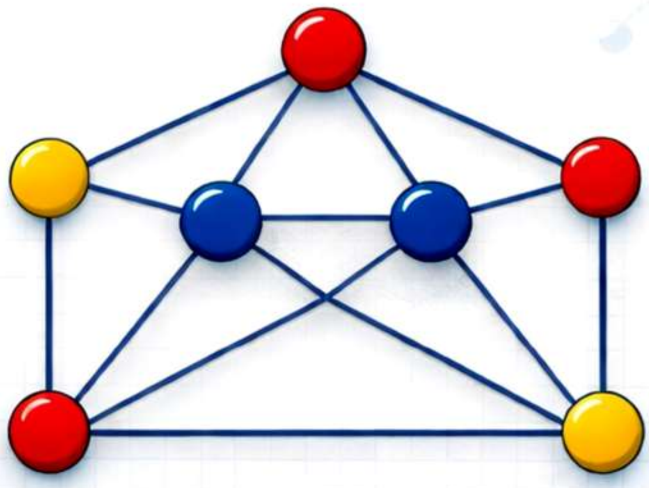

BY

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GRAPH THEORY

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GRAPH THEORY

OBJECTIVES:

- To introduce the concept of Graphs.
- To provide a sound knowledge on Trees and Spanning Trees.
- To gain knowledge about Matrices of Graphs and Digraphs.

Course Outline

Unit I: Introduction – Application of Graphs – Finite and Infinite graphs – Incidence and Degree – Isolated Vertex, Pendent Vertex and Null graph – Isomorphism – Subgraphs – Walks, Paths and Circuits – Connected graphs – Disconnected Graphs and Components.

Unit II: Euler graphs – Operations on Graphs – More on Euler graphs – Hamiltonian Paths and Circuits - Trees – Some properties on Trees – Pendent vertices in a tree - Distance and centres in a Tree – Spanning Trees.

Unit III: Incidence Matrix – Circuit Matrix – Fundamental Circuit Matrix and Rank of B – Path Matrix – Adjacency Matrix.

Unit IV: Planar Graphs – Kuratowski's two graphs – Euler's formula – Chromatic Number – Chromatic Partitioning – Chromatic polynomial.

Unit V: Matching – Coverings – Four colour Problem – Definition – Some types of Digraphs – Directed paths and connectedness – Euler Digraphs.

Text Book	**Graph Theory with Applications to Engineering & Computer Science, *“NARSINGH DEO”*,** Dover Publications, INC. Mineola, New York.
Unit I	**Chapter 1:** Sec. 1.1 to 1.5 and **Chapter 2:** Sec. 2.1, 2.2, 2.4, 2.5
Unit II	**Chapter 2:** Sec. 2.6 to 2.9 and **Chapter 3:** Sec. 3.1 to 3.4, 3.7
Unit III	**Chapter 7:** Sec. 7.1, 7.3, 7.4, 7.8, 7.9
Unit IV	**Chapter 5:** Sec. 5.2 to 5.4 and **Chapter 8:** Sec. 8.1 to 8.3
Unit V	**Chapter 8:** Sec. 8.4 to 8.6 and **Chapter 9:** Sec. 9.1, 9.2, 9.4, 9.5

UNIT – I

Introduction

1.1WHAT IS A GRAPH?

A graph (or simply a graph) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, v_3, \ldots\}$ called vertices and another set $E = \{e_1, e_2, e_3, \ldots\}$ whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i and v_j associated with edges e_k are called the end vertices of e_k.

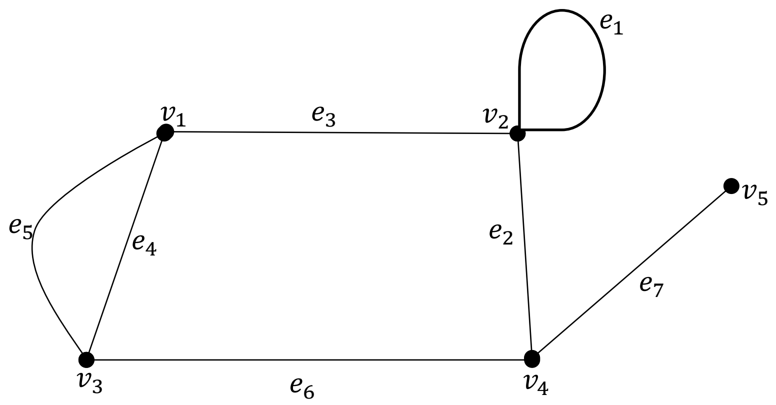


Fig 1.1

Definition 1.1:

An edge having the same vertex as both its end vertices is called a **Self-loop** (or loop). For example Fig 1.1, the edge e_1 is a self-loop.

Definition 1.2:

A graph allows more than one edge associated with a given pair of vertices are called the **Parallel edges**. For example Fig 1.1, the edge e_4 and e_5 is an parallel edges.

Definition 1.3:

A graph that has neither self-loops nor parallel edges is called a **Simple graph**.

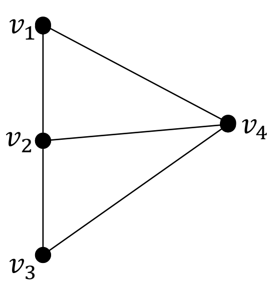


Fig 1.2 (a)

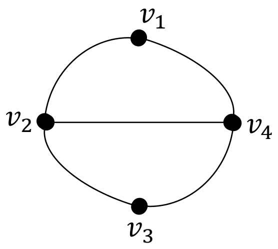


Fig 1.2 (b)

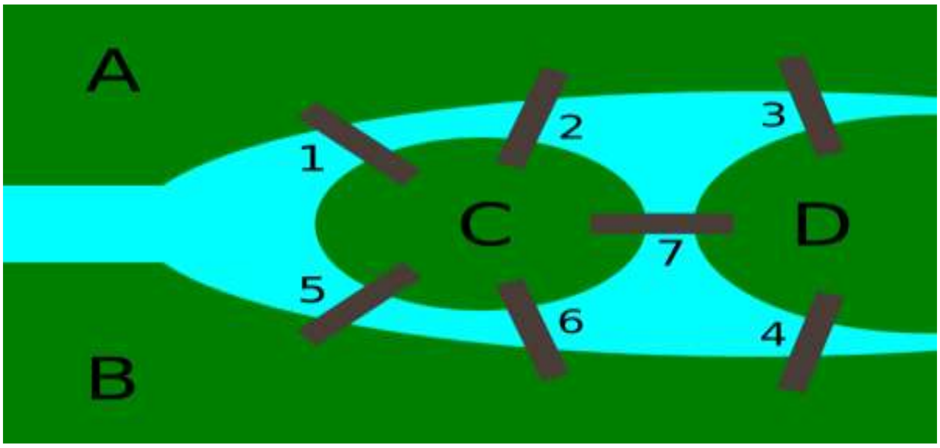
Note:

- The above two graphs drawn are the same, because incidence between edges and vertices is the same in both cases.
- A graph is also called a linear complex, a 1-complex, or a one-dimensional complex.
- A vertex is also referred to as a node, a junction, a point, 0-cell or an 0-simplex.
- An edge are a branch, a line, an element, a 1-cell, an arc and a 1- simplex.
- Generally, use the terms graph, vertex and edge.

Applications of Graphs

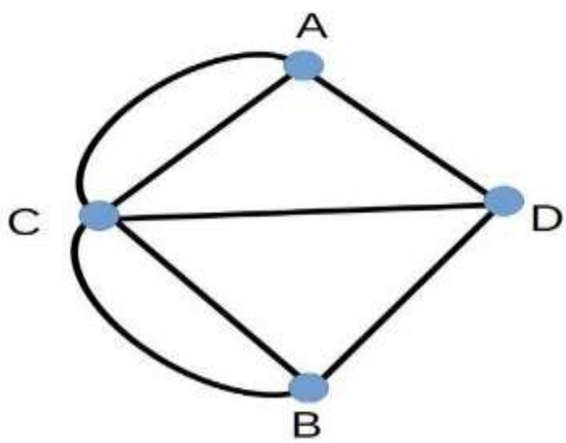
1. KONIGSBERG BRIDGE PROBLEM:

The Konigsberg bridge problem is perhaps the best known example in graph theory. It was a long-standing problem until solved by Leonhard Euler in 1736, by mean of a graph. In this problem was originated of the graph theory.



Two islands C and D formed by the pregel River in Konigsberg were connected to each other and to the banks A and B with seven bridges. The problem was to start at any of the four land areas of the city A, B, C or D walk over each of the seven bridge exactly once and return to the starting point (without swimming across the river of course).

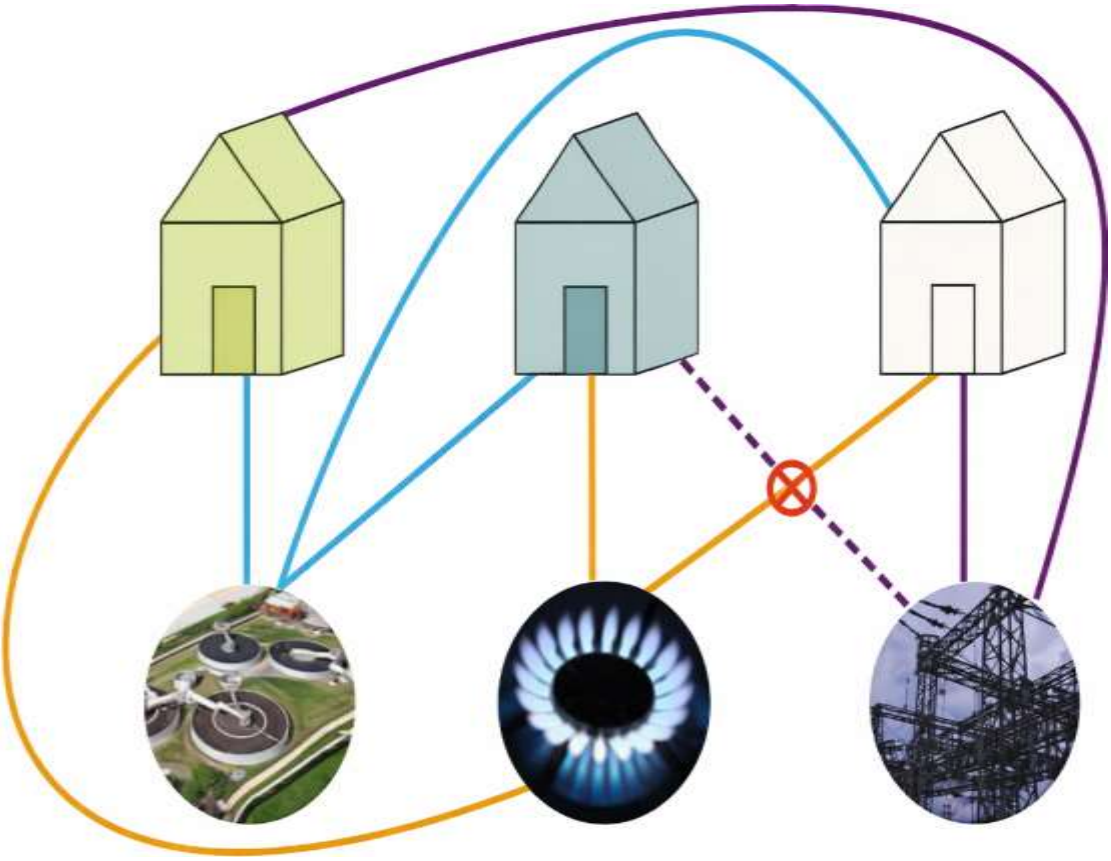
Euler represented this situation by means of a graph. The vertices represent the land areas and the edges represent the bridges.



Euler proved that a solution for this problem does not exist.

2. UTILITIES PROBLEM:

There are three houses H_1, H_2 and H_3 each to be connected to each of the three utilities water (W), gas (G) and electricity (E) by means of conduits?



This problem can be represented by a graph the conduits are shown as edges while the houses and utility supply centres are vertices. As we shall the graph cannot be drawn in the plane without edges crossing over. Thus, the solution of this problem does not exist.

3. ELECTRICAL NETWORK PROBLEM:

An electrical network are functions of only two factors.

i) The nature and value of the elements forming the network, such as resistors, inductors, transistors and so forth.

ii) The way these elements are connected together, that is the topology of the network.

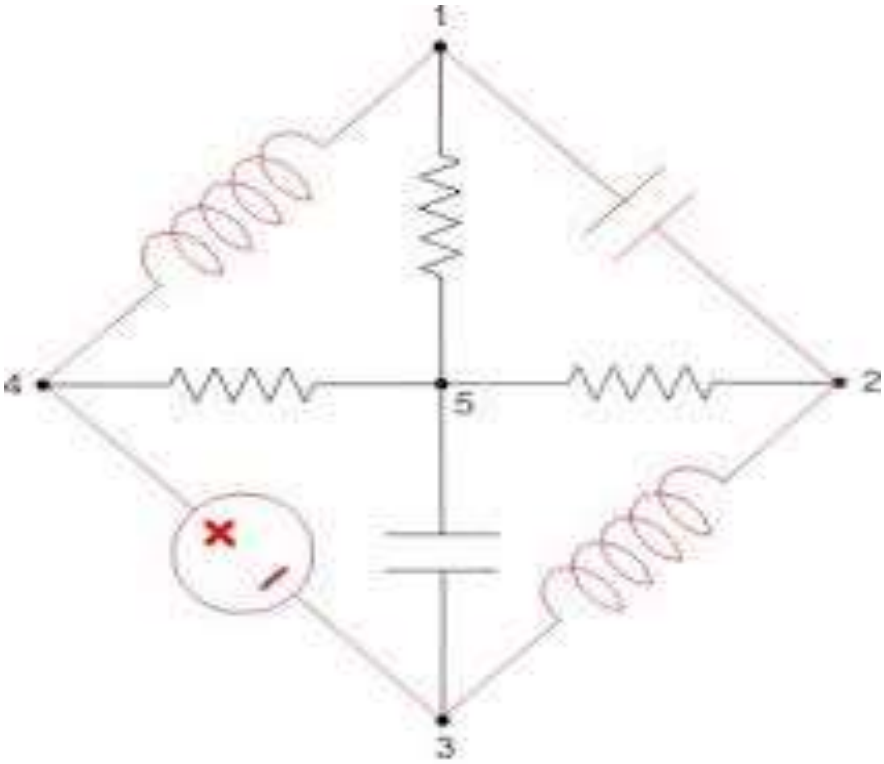
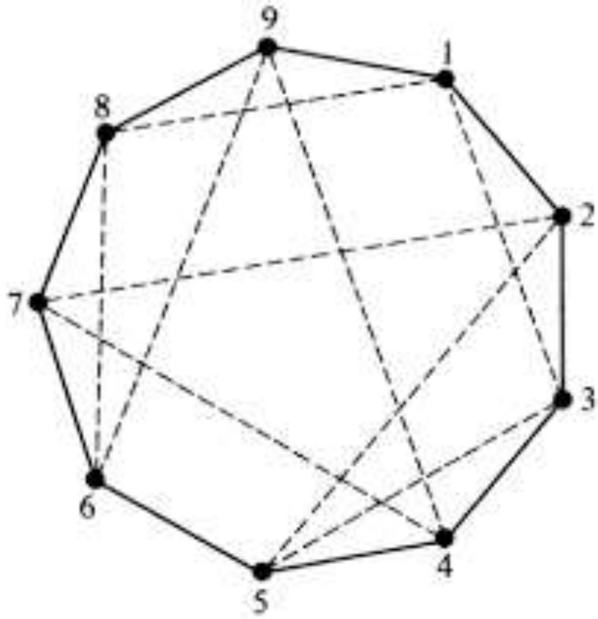


Figure-5

Since there are only a few different types of electrical elements, the variations in networks are chiefly due to the variations in topology. Thus electrical network analysis and synthesis are mainly the study of network topology. In the topological study of electrical network, factor 2 is separated from 1 and is studied independently. The advantage of this approach will be clearer in devoted solely to applying graph theory to electrical networks.

4.SEATING PROBLEM:

This situation can be represented by a graph with nine vertices such tat each vertex represents a member and an edge joining two vertices represents the relationship of sitting next to each other.



The two possible seating arrangements. These are 1 2 3 4 5 6 7 8 9 1 (solid lines) and 1 3 5 2 7 4 9 6 8 1 (dashed lines). It can be shown by graph-theoretic considerations that there are only two more arrangements possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is

$$\frac{n-1}{2} \text{ if } n \text{ is odd}$$

$$\frac{n-2}{2} \text{ if } n \text{ is even.}$$

Definition 1.4:

A graph with finite number of vertices as well as a finite number of edges is called a **finite** graph. Otherwise, it is an **infinite** graph.

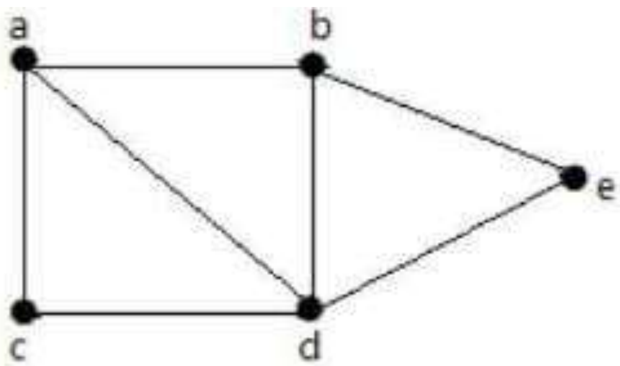


Fig 1.3: Finite Graph

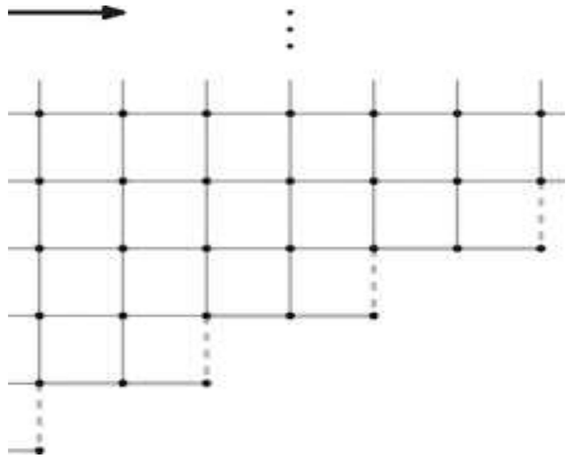


Fig 1.4: Infinite Graph

Definition 1.5:

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be **incident** with each other. For example, edges e_2, e_6 and e_7 are incident with vertex v_4.

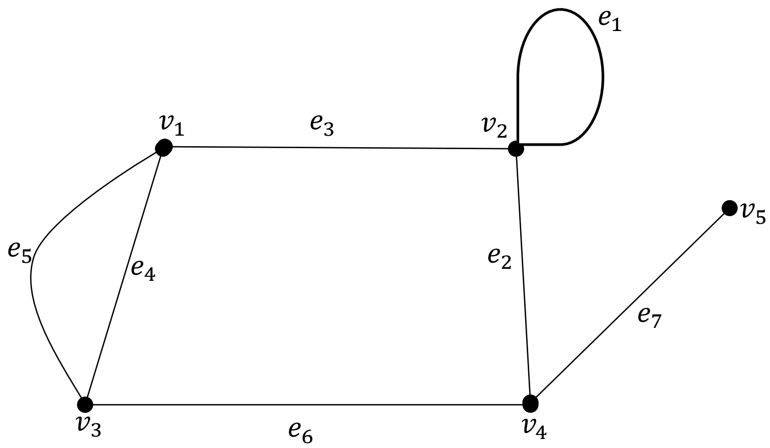


Fig 1.5

Definition 1.6:

Two nonparallel edges are said to be **adjacent** if they are incident on a common vertex. In Fig 1.5, e_2 and e_7 are adjacent of the edges.

Definition 1.7:

Two vertices are said to be adjacent if they are the end vertices of the same edge. In Fig 1.5, v_4 and v_5 are adjacent, but v_1 and v_4 are not adjacent.

Definition 1.8:

The number of edges incident on a vertex v_i with self-loops counted twice is called the **degree of the vertex** v_i. It is denoted by $d(v_i)$. In Fig 1.5 $d(v_1) = d(v_3) = d(v_4) = 3$, $d(v_2) = 4$ and $d(v_5) = 1$. The degree of vertex is sometimes also referred to as its **valency**.

Remark:

Let us now consider a graph G with e edges and n vertices $v_1, v_2, \dots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G.

(ie) $\sum_{i=1}^{n} d(v_i) = 2e$ ----------------------- (1)

For example taking the Fig 1.5,

$$\sum_{i=1}^{n} d(v_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$$

$$= 3 + 4 + 3 + 3 + 1$$

$$= 14$$

$$= 2(7)$$

$=$ twice the number of edges.

Theorem 1.1:

The number of vertices of odd degree in a graph is always even.

Proof:

Let us consider the vertices with odd and even degree separately, the quantity in the left sides of Eq.(1) can be expressed as the sum of two sums each taken over vertices of even and odd degrees respectively as follows,

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k) \qquad \text{----------------------- (2)}$$

Since the left-hand side in Eq.(2) is even and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even

$$\sum_{odd} d(v_k) = \text{an even number} \qquad \text{-------------------------(3)}$$

Because in Eq.(3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

Definition 1.9:

A graph in which all vertices are of equal degree is called a **regular graph**.

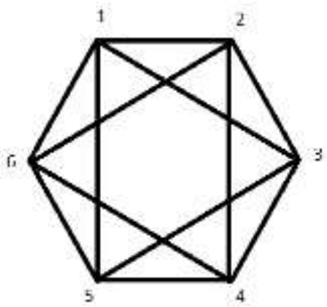


Fig 1.6

Definition 1.10:

A vertex having no incident edge is called an **isolated vertex**. (vertices of degree is zero). A vertex of degree one is called a **pendant vertex** or an end vertex.

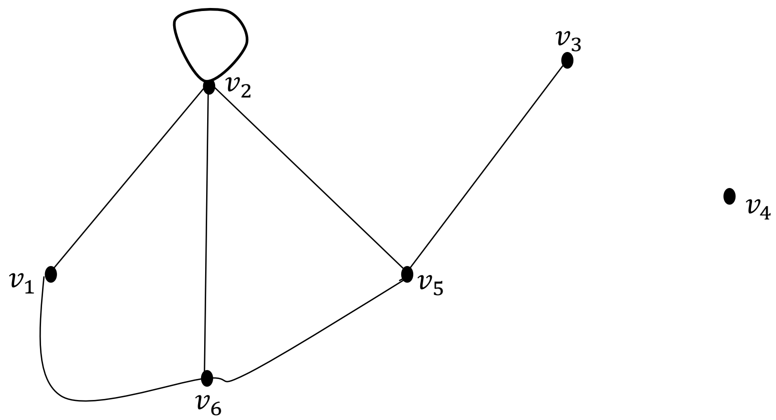


Fig 1.7

In Fig 1.7, the vertex v_4 is an isolated vertex and v_3 is an pendant vertex.

Definition 1.11:

A graph $G = (V, E)$ is is possible for the edge set E to be empty. Such a graph without any edges is called a **null graph**.

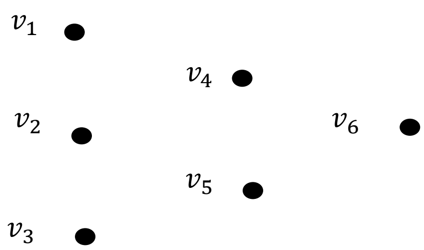
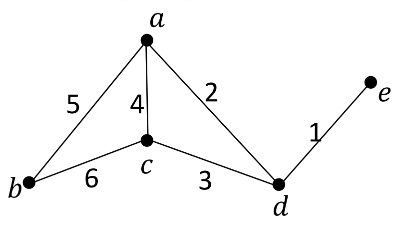


Fig 1.8

Definition 1.12:

Two graphs G and G' are said to be **isomorphic**, if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.



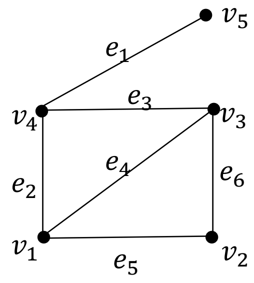


Fig 1.9

Remark:

The isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges
3. An equal number of vertices with a given degree,

Definition 1.13:

A graph g is said to be a subgraph of a graph G if all the vertices and the edges of g are in G and each edge of g has the same end vertices in g as in G.

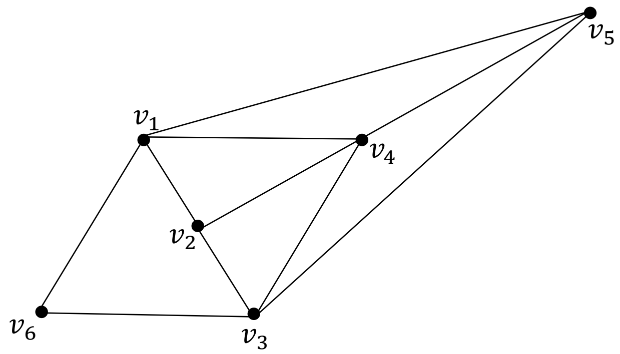


Fig 1.10 (a)

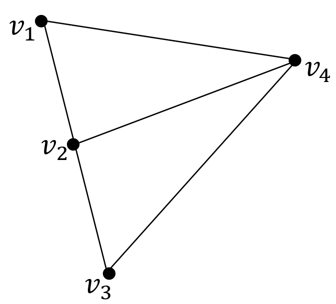


Fig 1.10 (b)

In a graph Fig 1.10 (a) and one of its subgraphs Fig 1.10 (b)

Observation:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices is also a subgraph of G.

Definition 1.14:

Two (or more) subgraphs g_1 and g_2 of a graph G are said to be **edge disjoint** if g_1 and g_2 do bot have any edges in common.

Definition 1.15:

Subgraphs that do not even have vertices in common are said to be **vertex disjoint**. (ie, Graphs that have no vertices in common cannot possibly have edges in common)

Note:

Edge-disjoint graphs do not have any edge in common, they may have vertices in common.

Definition 1.16:

A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk.

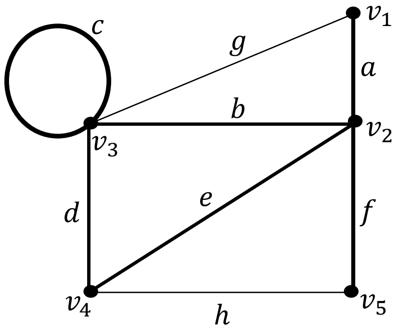


Fig 1.11

In Fig 1.11, $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk .

Definition 1.17:

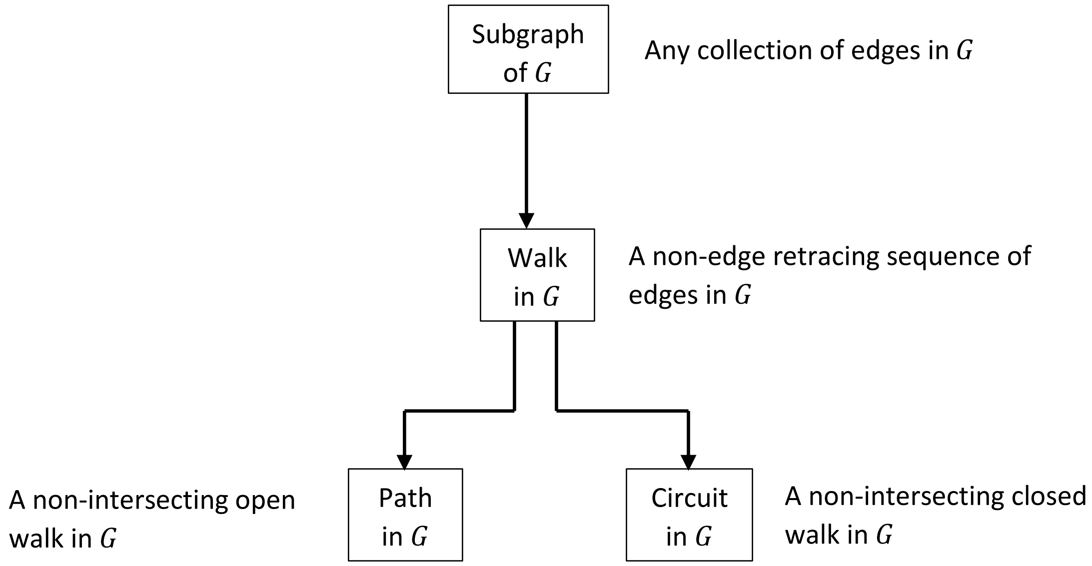
A walk to begin and end at the same vertex, such a walk is called a **closed walk**. A walk that is not closed for a begin and end vertices is called an **open walk**.

Definition 1.18:

An **open walk** in which no vertex appears more than once is called a **path**. In Fig 1.11, $v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path.

Definition 1.19:

A **closed walk** in which no vertex (expect the initial and the final vertex) appears more than once is called a **circuit**. In Fig 1.11, $v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is a circuit.



Definition 1.20:

A graph G is said to be connected if there is at least one path between every pair of vertices in G. Otherwise, G is disconnected. In Fig 1.11, is a connected graph and Fig 1.8 is a disconnected graph.

Definition 1.21:

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a **component**. In Fig 1.7, consist of two components.

Theorem 1.2:

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof:

Suppose that such a partitioning exists.

Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b.

Otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a.

Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will from a set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition.

Theorem 1.3:

If a graph (connected or disconnected) has exactly two vertices of odd degree there must be a path joining these two vertices.

Proof:

Let G be a graph that has exactly two vertices of odd degree. Let these vertices be u and v.

Suppose u and v are not connected by any path. Then, u and v must belong to different components of the graph.

Consider the component containing u. In this component, u has odd degree.

From, the handshaking lemma, the number of vertices with odd degree any graph (or component) must be even.

Therefore, the component containing u must contain another vertex of odd degree.

Similarly, the component containing v must also contain another vertex of odd degree. Thus, the graph would have at least four vertices of odd degree.

This is the contradiction, the given condition that the graph has exactly two vertices of odd degree. Therefore, our assumption is wrong. Hence, there must exist a path joining the two odd degree vertices u and v.

Therefore, if a graph has exactly two vertices of odd degree, they are connected by a path.

Theorem 1.4:

A simple graph with $\boldsymbol{n}$ vertices and $\boldsymbol{k}$ components can have at most $\boldsymbol{(n-k)(n-k+1)/2}$ edges.

Proof:

Let G be a simple graph with n vertices and k components.

Assume the components contain $n_1, n_2, n_3, \dots, n_k$ vertices respectively.

Thus, $n_1 + n_2 + n_3 + \cdots + n_k = n$

Since, the graph is simple the maximum number of edges in a component with n_i vertices is when the component is a complete graph.

Hence the maximum number of edges in the i^{th} component of G is $\frac{n_i(n_i-1)}{2}$.

Therefore, the total number of edges in G is at most $m \leq \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$.

$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2}$$

$$= \frac{1}{2}[n^2 - 2nk + k^2 + 2n - k - n]$$

$$= \frac{1}{2}[n^2 - 2nk + k^2 + n - k]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence proved.

UNIT - II

EULER GRAPH

Definition 2.1:

An **Euler Graph** is a connected graph in which all vertices have even degree. Such a graph contains an Euler Circuit, which is a closed path that begin and end at the same vertex and traverse every edge exactly once.

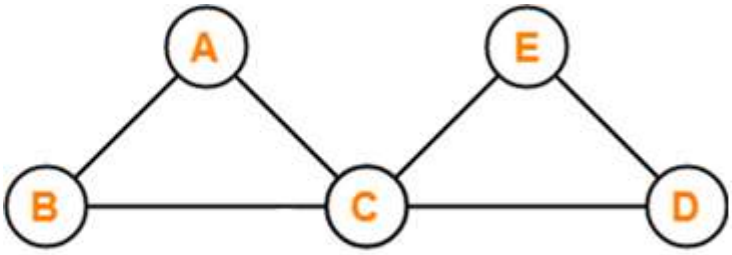


Fig 2.1

Theorem 2.1:

A connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof:

We prove the theorem in **two parts**.

($\Rightarrow$) If G is an Euler graph, then every vertex has even degree.

Assume that G is an Euler graph. Then G contains an Euler Circuit, which is a closed trail that starts and ends at the same vertex and uses every edge exactly once.

Whenever the circuit enters a vertex through one edge, it must leave through another edge. Thus, the edges at each vertex occur in pairs (one entering and one leaving).

Therefore, the degree of every vertex must be even.

(⇐) If every vertex of GGG has even degree, then GGG is an Euler graph.

Assume G is connected and every vertex has even degree. Start from any vertex and follow edges one by one without repeating any edge until returning to the starting vertex.

Since every vertex has even degree, whenever we reach a vertex there will always be another unused edge to leave it.

Thus we obtain a closed trail. If this trail contains all edges of GGG, then it is an Euler circuit.

If some edges are left, we start another trail from a vertex of the existing trail and combine them. Continuing this process, eventually all edges are included in one closed trail.

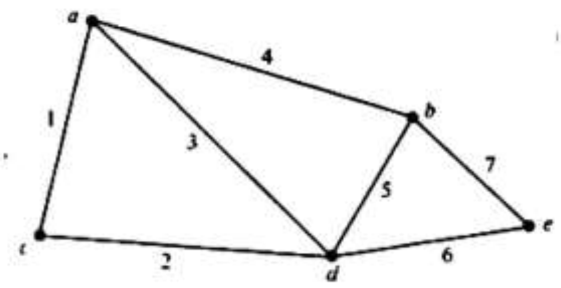
Hence G has an Euler circuit, so G is an Euler graph.

Therefore, a connected graph is Eulerian if and only if every vertex has e ven degree.

Hence proved.

Definition 2.2:

A **Unicursal Graph** is a connected graph in which exactly two vertices have odd degree and all other vertices have even degree.



Unicursal graph with a walk a 1 c 2 d 3 a 4 b 5 d 6 e 7 b.

Fig 2.2

Theorem 2.2:

In a connected graph G with exactly $2k$ odd vertices, there exist k edge – disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Proof:

Since G has exactly $2k$ odd vertices, we can pair them into k disjoint pairs of odd vertices $(u_1, v_1), (u_2, v_2), \dots, (u_k, v_k)$.

Now, for each pair (u_i, v_i) add an extra edge between u_i and v_i.

$$(u_1 - v_1) - (u_2 - v_2) - (u_k - v_k)$$

This gives a new graph G' where all vertices are now even.

Since G' is Eulerian, it has an Eulerian circuit, say C.

Now, we can decompose C' into k edge-disjoint circuits $C_1, C_2, \dots, C_k$ such that

- each C_i is an Eulerian circuit.
- C_i contains the edge (u_i, v_i).
- $C_1, C_2, \dots, C_k$ cover all edges of G'.

Finally, removing extra edges (u_i, v_i) from each C_i, we obtain k edge-disjoint Unicursal subgraph of G.

Hence proved.

OPERATIONS ON GRAPHS:

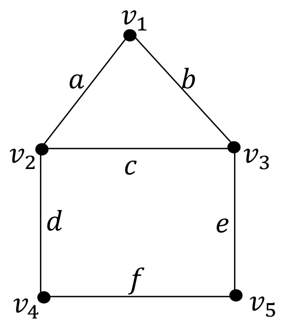
In graph theory, **operations on graphs** are mathematical processes used to create new graphs from one or more initial ones. These operations are fundamental for modeling complex systems, such as chemical molecules, social networks, or communication grids, by building them from simpler components. Operations are generally classified into two main categories based on the number of input graphs:

Unary Operations: These create a new graph from a single initial graph by making local or global structural changes.

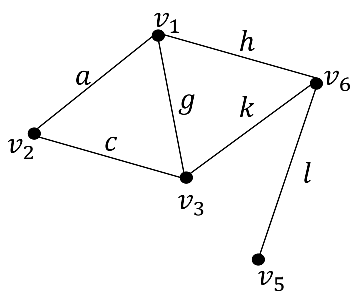
Binary Operations: These combine two initial graphs $G_1$and G_2 to produce a composite graph.

Definition 2.3:

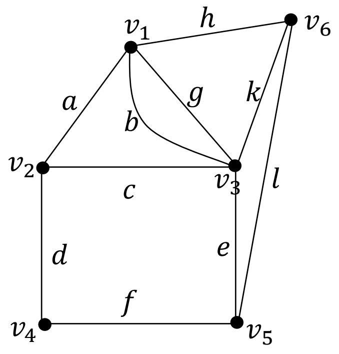
The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \cup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$. Likewise, the **intersection** $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2. The **ring sum** of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but not in both.



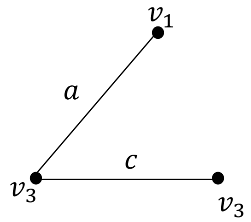
G_1



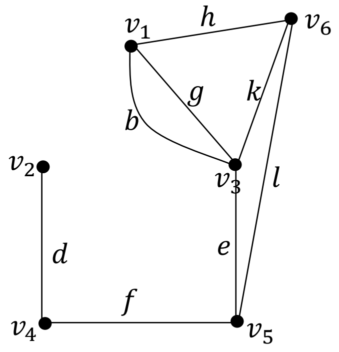
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \oplus G_2$

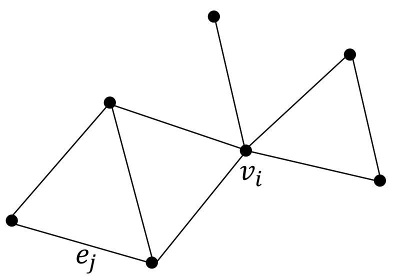
Fig 2.3

Definition 2.4:

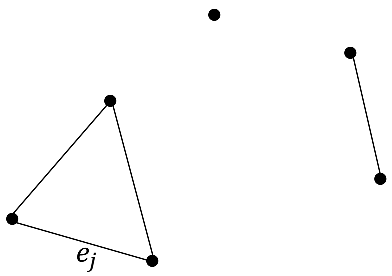
A graph G is said to have been decomposed into two subgraphs g_1 and g_2 if $g_1 \cup g_2 = G$ and $g_1 \cap g_2 =$ a null graph.

Definition 2.5:

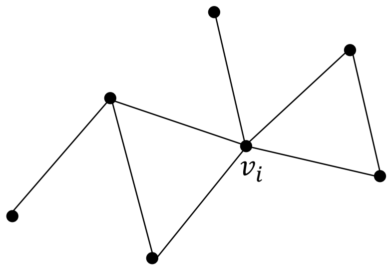
If v_1 is a vertex in graph G, the $G - v_1$ denotes a subgraph of G obtained by deleting v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices. Therefore $G - e_j = G \oplus e_j$.



G



$G - v_j$



$G - e_j$

Fig 2.4

Definition 2.6:

A pair of vertices a, b in a graph are said to be fused if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus **fusion** of two vertices does not alter the number of edges but it reduces the number of vertices by one.

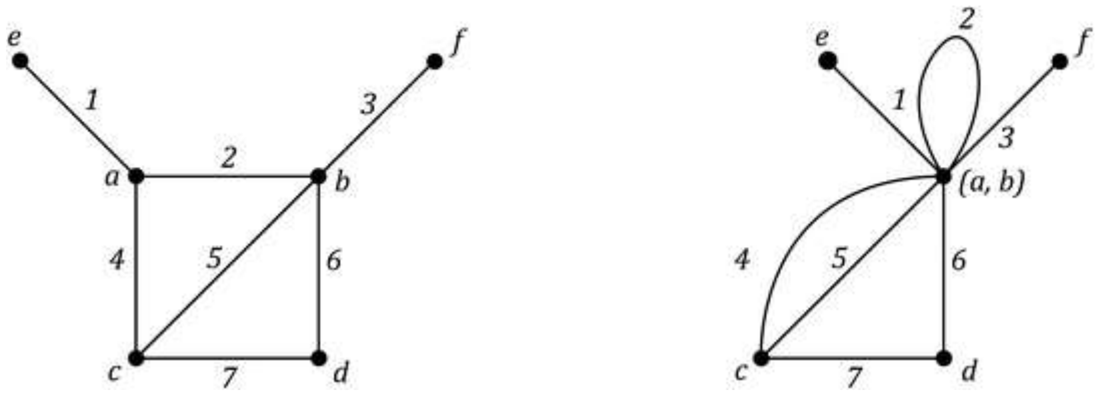


Fig 2.5

Theorem 2.3:

A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Proof:

⇒ If G is an Euler graph, then it can be decomposed into circuits.

Suppose G is an Euler graph.

By definition of an Euler graph, there exists an Euler circuit in G, that is, a closed trail which traverses every edge of G exactly once.

While traversing the Euler circuit, whenever a vertex is repeated, the edges between two successive appearances of that vertex form a circuit.

Thus the Euler circuit can be split into several smaller edge–disjoint circuits.

These circuits together contain all edges of G.

Therefore, the graph G can be decomposed into circuits.

⇐ If G can be decomposed into circuits, then G is an Euler graph.

Suppose the graph G can be decomposed into a set of circuits

$$C_1, C_2, C_3, \dots , C_k$$

such that every edge of G belongs to exactly one of these circuits.

In every circuit, whenever a vertex is entered, it is also left.
Thus each circuit contributes two edges at each visited vertex.

Hence every vertex has even degree.

Therefore all vertices of G are of even degree.

Since G is connected and every vertex has even degree, G contains an Euler circuit.

Thus G is an Euler graph.

Definition 2.7:

A graph G is said to be **arbitrarily traceable** if every path in G is contained in a Hamiltonian path of G**.**

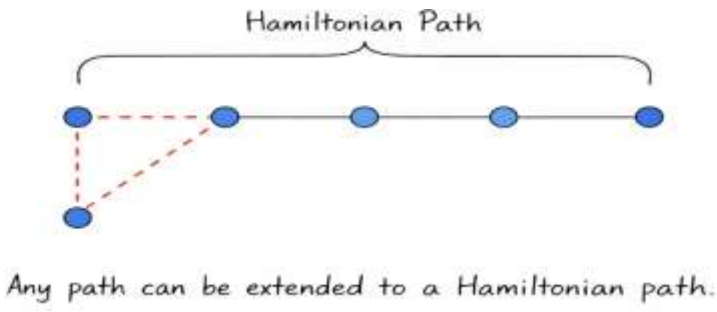


Fig 2.6

Note:

- A path is a sequence of vertices where no vertex is repeated.
- A Hamiltonian path is a path that visits all vertices of the graph exactly once.
- In an arbitrarily traceable graph, any path you choose can always be extended until it becomes a Hamiltonian path.

Definition 2.8:

A **Hamiltonian path** in a graph is a path that visits every vertex of the graph exactly once**.**

- Each vertex appears only once in the path.

➢ The path does not need to return to the starting vertex.

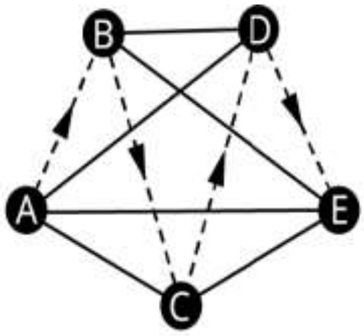


Fig 2.7

Definition 2.9:

If a Hamiltonian path starts and ends at the same vertex is called a **Hamiltonian cycle (or Hamiltonian circuit)**.

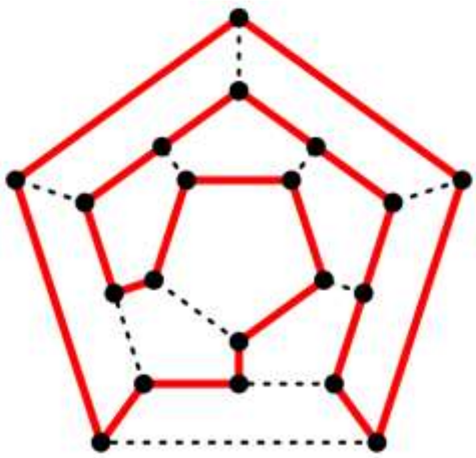


Fig 2.8

Definition 2.10:

A **complete graph** is a simple graph in which every pair of distinct vertices is connected by exactly one edge**.** In other words, each vertex is adjacent to every other vertex in the graph**.** It is denoted by K_n.

Properties:

1. Number of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

2. Degree of each vertex is $n - 1$.

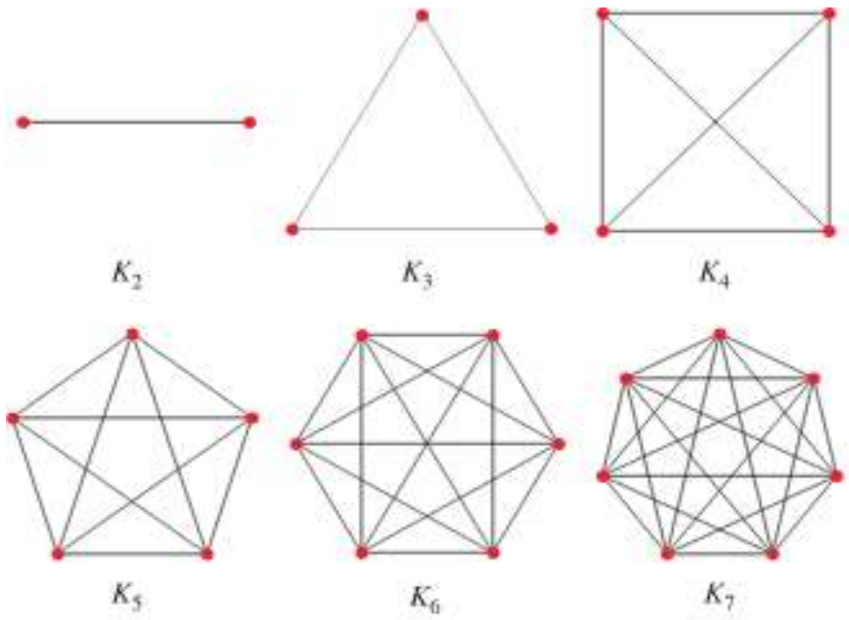


Fig 2.9

Theorem 2.4:

In a complete graph with n vertices there are $\frac{n-1}{2}$ edge-disjoint Hamiltonian circuits if n is an odd number ≥ 3.

Proof:

Let n be an odd integer with $n \geq 3$.

Let the vertices of the complete graph K_n be $v_1, v_2, v_3, \dots, v_n$.

Since K_n is complete every pair of vertices is connected by an edge.

A Hamiltonian circuit in a graph with n vertices contains exactly n edges because it visits every vertex once and returns to the starting vertex.

If the graph is decomposed into edge-disjoint Hamiltonian circuits, each circuits uses n edges.

Thus, the number of such circuits $\frac{n(n-1)}{2} = \frac{n-1}{n} = \frac{n-1}{2}$

Arrange the vertices of K_n on a circuit. Fix one vertex and rotate the remaining vertices around it to form different circuits.

Each rotation can produce the Hamiltonian circuit and one no edge is repeated.

Thus, we obtain exactly $\frac{n-1}{2}$ edge-disjoint Hamiltonian circuits.

Hence Proved.

TREES AND FUNDAMENTAL CIRCUITS

Definition 2.11:

A **tree** is a connected graph without any circuits. Trees with one, two, three and four vertices are shown in Fig 2.10.

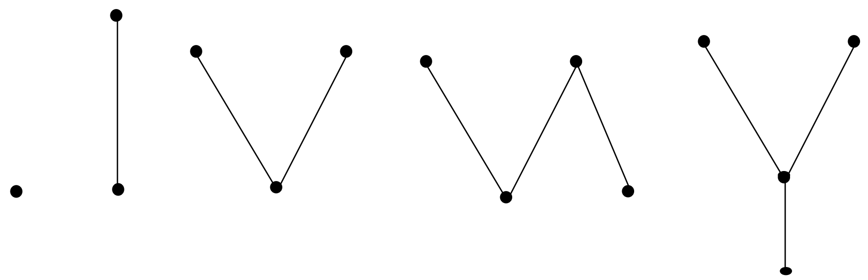


Fig 2.10

SOME PROPERTIES OF TREES

Theorem 2.5:

There is one and only one path between every pair of vertices in a tree T.

Proof:

Let u and v be any two distinct vertices in the tree T.

Existence of a path

Since T is a connected graph, there must exist at least one path between any two vertices u and v.

So, a path between u and v exists.

Uniqueness of the path

Assume that there are two different paths between u and v.

Let these paths be:

- P_1 : a path from u to v
- P_2 : another path from u to v

Since P_1 and P_2 are different, they must differ at some vertex or edge. If we combine these two distinct paths, they will form a cycle in the graph. But a tree is defined as a connected graph with no cycles. Thus, the existence of two different paths would create a cycle**,** which contradicts the definition of a tree**.**

Therefore, our assumption is wrong.

Hence, there cannot be more than one path between two vertices.

Since a path exists and cannot be more than one, there is exactly one path between every pair of vertices in T. Hence, there is one and only one path between every pair of vertices in a tree.

Hence Proved.

Theorem 2.6:

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof:

To prove that G is a tree, we must show that G is connected and contains no cycles.

Claim 1: G is Connected

Given that there is exactly one path between every pair of vertices in G.

Therefore, for any two vertices u and v in G, a path exists between them.

Hence, G is connected.

Claim 2: G has no cycles

Assume that G contains a cycle.

Let u and v be two vertices on this cycle.

In a cycle, there are at least two distinct paths between u and v (one going in each direction around the cycle).

Thus, there would be more than one path between u and v.

This contradicts the given condition that there is exactly one path between every pair of vertices.

Therefore, G cannot contain a cycle. Hence, G is acyclic. Since G is connected and contains no cycles, G is a tree.

Therefore, if there is exactly one path between every pair of vertices in G, then G is a tree.

Hence Proved.

Theorem 2.7:

A tree with n vertices has $n - 1$ edges.

Proof:

We prove using mathematical induction.

Consider a tree with $n = 1$ vertex.

Such a graph contains no edges.

Thus, Number of edges$= 0 = 1 - 1$

Hence, the statement is true for $n = 1$.

Induction Hypothesis

Assume that every tree with k vertices has $k - 1$ edges**.**

Consider a tree T with $k + 1$ vertices.

A property of trees is that every tree has at least one pendant (leaf) vertex, that is, a vertex of degree 1**.**

Let v be a pendant vertex and let e be the edge incident with v.

Remove the vertex v and the edge e from the tree T.

The remaining graph is still a tree and it has k vertices**.**

By the **induction hypothesis**, this tree has $k - 1$ edges

Now add back the removed vertex v and the edge e.

Thus, the number of edges becomes

$$(k - 1) + 1 = k$$

Since the number of vertices is $k + 1$, the number of edges is

$$k = (k + 1) - 1$$

Therefore, by **mathematical** induction, a tree with n vertices has $n - 1$ edges.

Hence Proved.

Theorem 2.8:

Any connected graph with n vertices and $n - 1$ edges is a tree.

Proof:

Let G be a connected graph with n vertices and $n - 1$ edges.

To prove that G is a tree, we must show that G has no cycles.

Assume the contrary

Assume that G contains a cycle. Remove one edge from this cycle.

After removing the edge:

- The graph will still remain connected, because the remaining edges of the cycle still connect the vertices.
- The number of edges becomes $n - 2$.

Continue removing cycle edges. If cycles still exist, remove one edge from each cycle while keeping the graph connected. Finally, we obtain a connected graph with no cycles. Such a graph is a tree.

Use the property of trees

A tree with n vertices has $n - 1$ edges. But after removing an edge from the cycle, the graph has $n - 2$ edges while still being connected.

This contradicts the property that a connected acyclic graph with n vertices must have $n - 1$ edges.

Therefore, our assumption that G contains a cycle is false.

Hence, G contains no cycles. Since G is connected and acyclic, G is a tree.

Therefore, any connected graph with n vertices and $n-1$ edges is a tree.

Hence Proved.

Theorem 2.9:

A graph is a tree if and only if it is minimally connected.

Proof:

Necessary part:

If G is a tree, then it is minimally connected

Let G be a tree.

By definition, a tree is a connected graph with no cycles.

Consider any edge e of G. If, we remove e, the graph becomes disconnected. This is because in a tree there is exactly one path between every pair of vertices. Removing the edge e breaks this unique path.

Thus, removing any edge disconnects the graph.

Therefore, G is minimally connected.

Sufficient part:

If G is minimally connected, then it is a tree

Let G be a minimally connected graph, meaning:

- G is connected, and
- Removing any edge makes it disconnected.

We must show that G has no cycles.

Assume that G contains a cycle. Let e be any edge in that cycle. If, we remove e, the remaining edges of the cycle still connect the vertices. Thus the graph remains connected**,** which contradicts the assumption that G is minimally connected.

Therefore, G cannot contain a cycle**.** Hence G is acyclic. Since G is connected and acyclic**,** G is a tree**.**

Hence Proved.

Theorem 2.10:

A graph G with n vertices $n-1$ edges and no circuits is connected.

Proof:

We prove this contradiction.

Assume that G is not connected. Then G consists of $k \geq 2$ connected components.

Let these components be $G_1, G_2, G_3, \dots, G_k$.

Let $n_i =$ number of vertices in component G_i.

So, $n_1 + n_2 + n_3 + \cdots + n_k = n$.

Since, G has no circuits each component G_i is also acyclic.

Thus, each G_i is a tree.

We known that, a tree with n_i vertices has exactly $n_i - 1$ edges. So, total number of edges $G_i$is $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)$.

$= (n_1 + n_2 + n_3 + \cdots + n_k) - k$

$= n - k$ ----------------- (1)

Here, if G is a tree, them number of edges in $G = n - 1$ ----------------- (2)

Comparing the (1) and (2), we get

$n - k = n - 1$

$k = 1$, there is only one connected component.

Therefore, G is connected. Hence, a graph with n vertices, $n - 1$ edges and no circuits is connected.

Hence Proved.

Results:

The results of the preceding six theorems can be summarized by saying that the following are five different but equivalent definitions of a tree. That is, a graph G with n vertices is called a tree if

- G is connected and is circuitless, or
- G is connected and has $n - 1$ edges, or
- G is circuitless and has $n - 1$ edges, or
- There is exactly one path between every pair of vertices in G, or

➢ G is a minimally connected graph.

PENDANT VERTICES IN A TREE

Definition 2.12:

A **pendant** vertex was defined as a vertex of degree one.

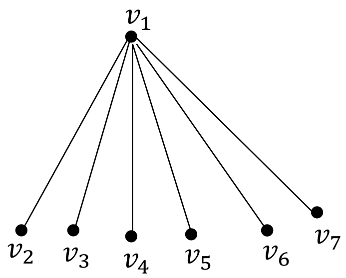


Fig 2.11

Theorem 2.11:

In any tree (with two or more vertices), there are at least two pendant vertices.

Proof:

Let T be a tree with $n \geq 2$ vertices.

Since T is a tree, it is connected and has no cycles**.**

Consider a longest path in the tree.

Take a longest path in T, say $P = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$

This means there is no path in T longer than this one.

To show that the endpoints are pendant vertices.

We claim that the vertices v_1 and v_k are pendant vertices (i.e., degree = 1).

To prove v_1:

Assume, for contradiction, that v_1 has degree greater than 1. Then there exists another vertex $u \neq v_2$ such that u is adjacent to v_1.

Now consider two cases:

- If u is not on the path P, then we can extend the path:

$$u \to v_1 \to v_2 \to \cdots \to v_k$$

  This is longer than P, which contradicts that P is the longest path.

- If u is on the path P, then a cycle is formed, which contradicts the fact that a tree has no cycles.

Thus, our assumption is false, and v_1 must have degree 1.

Similarly for v_k, by the same argument, v_k must also have degree 1. Thus, the tree has at least two vertices of degree 1, namely v_1 and v_k.

Hence Proved.

DISTANCE AND CENTERS IN A TREE

Definition 2.13:

In a connected graph G, the **distance** $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path between them. That is, the number of edges in the shortest path between them.

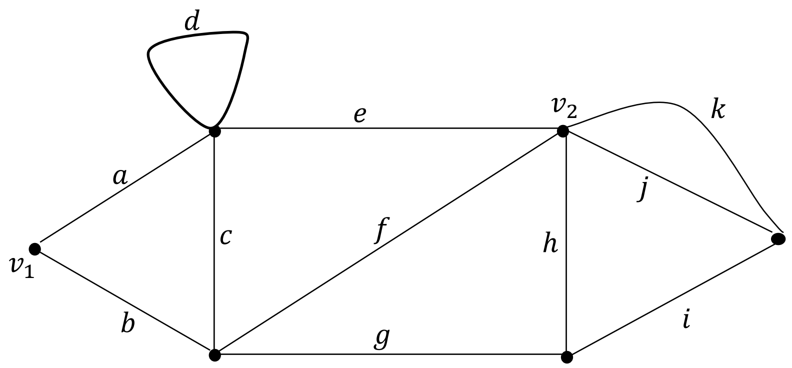


Fig: 2.12

Definition 2.14:

In a tree T, the **centers** of T is defined as the set of vertices with minimum eccentricity. That is, $Center\ T = \{v \in V(T)|\ e(v) is\ minimum\}$

Definition 2.15:

For a vertex v in T, the eccentricity of v denoted by $e(v) = \max\{d(u,v)|u \in V(T)\}$. That is, the maximum distance from v to any other vertex in the tree.

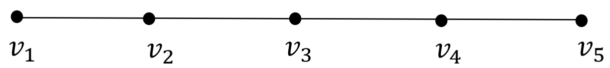


Fig: 2.13

For example in Fig:2.13, to find the eccentricity of a graph

$e(v_1) = 4$, $e(v_2) = 3$, $e(v_3) = 2$, $e(v_4) = 3$, $e(v_5) = 4$

Therefore, the minimum eccentricity is 2. Then, the graph of center is v_3.

PROPERTIES OF THE GRAPH METRIC:

1. **Non-negativity:** $d(u, v) \geq 0$
2. **Identity of Indiscernibles:** $d(u, v) = 0 \Leftrightarrow u = v$
3. **Symmetry:** $d(u, v) = d(v, u)$
4. **Triangle inequality:** $d(u, w) \leq d(u, v) + d(v, w)$ for any v.

Theorem 2.12:

The distance between vertices of a connected graph is a metric.

Proof:

Let G be a connected graph.
Define a function

$d(u, v) =$ length of the shortest path between vertices u and v.

We prove that d is a **metric** by verifying the four properties:

1. Non-negativity:

$$d(u, v) \geq 0$$

- The length of a path is the number of edges, which is always non-negative.
- Hence, $d(u, v) \geq 0$.

2. Identity of Indiscernibles:

$$d(u, v) = 0 \Leftrightarrow u = v$$

- The distance from a vertex to itself is 0 (no edges).
- If $u \neq v$, any path between them has at least one edge, so $d(u, v) > 0$.
- Hence, the property holds.

3. Symmetry:

$$d(u, v) = d(v, u)$$

- Any path from u to v can be traversed in reverse from v to u.
- Therefore, the shortest path length is the same in both directions.

4. Triangle Inequality:

$$d(u, w) \leq d(u, v) + d(v, w)$$

- Let the shortest path from u to v have length $d(u, v)$, and from v to w have length $d(v, w)$.
- Combining these gives a path from u to w of length $d(u, v) + d(v, w)$.
- Since $d(u, w)$ is the shortest path length, $d(u, w) \leq d(u, v) + d(v, w)$.

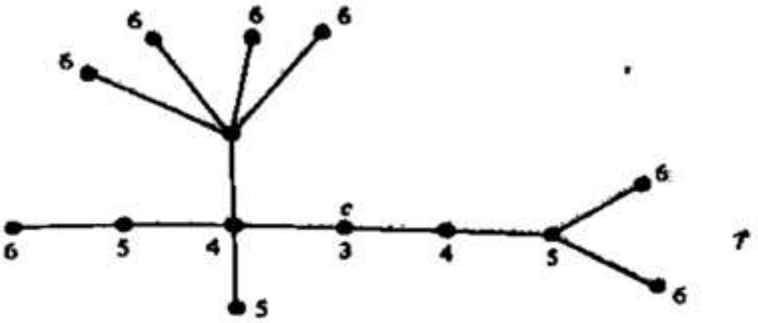
Hence Proved.

Theorem 2.13:

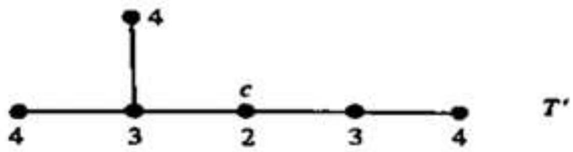
Every Tree has either one or two centers.

Proof:

Let T be a tree.



(a)



(b)

(c)

Fig: 2.14

(d)

Then, T has at least two pendant vertices.

Remove all the pendant vertices of T to obtain subtree T'

Since pendant vertices have maximum eccentricity they do not belong to the center.

Therefore, center of T remain in center of T' also.

Further, the eccentricity of each vertex of T which remains in T', is reduced uniformly by one.

Hence the vertex of the center of T have minimum eccentricityin T' also.

(ie) center of T = center of T'

We continue this process until we obtain a subtree $T_n$which is either k_1 or k_2.

Also, we have center of T = center of T' = … = center of $T^{n'}$

But $T^{n'}$ is either k_1 or k_2

Therefore, T has wither one or two centers.

Hence proved.

Corollary 2.14:

If a tree T has two centres, the two centers must be adjacent.

Proof:

We will use one observation that the maximum distance max $d(v,w)$ from a given vertex v to any other vertex w occurs only when w is pendant vertex.

Now, let T is a tree with n vertices $(n \geq 2)$

$\Rightarrow$ T must have atleast two pendant vertices.

Delete all pendant vertices from T, then resulting graph T’ is still a tree.

$\Rightarrow$ eccentricity $e(v)$ in T' is just one less than $e(v)$ in T $\forall$ v in T'

Again delete pendant vertices from T' so that resulting T'' is still a tree with same centers.

Note that all vertices that T had as centers will still remain centers in $T' \rightarrow T'' \rightarrow T''' \rightarrow \cdots$ continue this process untill remaining tree has either one vertex or one edge. So at the end, if one vertex is there this implies tree T has one center. If one edge is there then tree T has two centers.

Therefore, if a tree has two centers then these two centers must be adjacent.

Hence Proved.

Definition 2.16:

A tree T is said to be a **spanning tree** of a connected graph G if T is a subgraph of G and T contains all vertices of G.

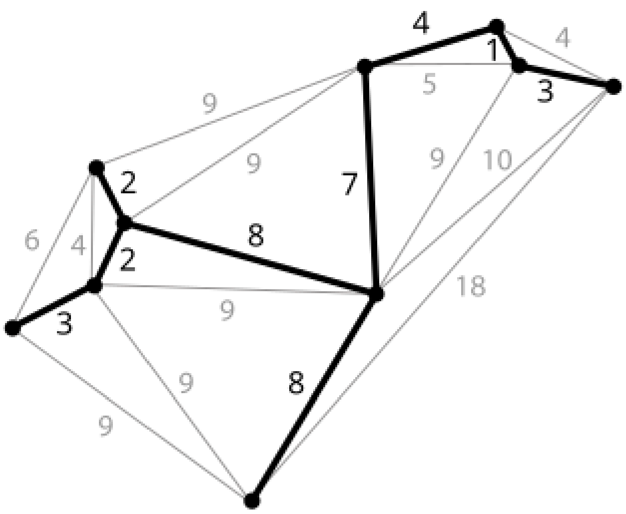


Fig: 2.15

Theorem 2.15:

Every connected graph has at least one spanning tree.

Proof:

Let G be any connected graph.

If G contaions no cycle, then G itself is a spanning tree.

Otherwise, if G contains a cycle C, let x be an edge of this cycle.

Now remove x from G and pbtain a spanning subgraph $G - x$ of G.

If $G - x$ does not contain any cycle, then $G - x$ is a spanning tree of G.

Otherwise, remove an edge y from a cycle C' in $G - x$ and obtain a spanning subgraph $G - x - y$ of G.

We continue this process until we obtain a spanning subgraph which has no cycle further.

Therefore, final graph is the required spanning tree of G.

Hence Proved.

Theorem 2.16:

With respect to any of its spanning trees, a connected graph of n vertices and e edges has $n - 1$ tree branches and $e - n + 1$ chords.

Definition 2.17:

The **rank** is the dimension of the subspace of the cycle space, or dimension of the incident matrix which equal $n-k$. Where n vertices and k components. It is denoted by $P(G)$.

(ie) $P(G) = n - k$

Definition 2.18:

It represent the number of fundamental cycles or the dimension of the cycle space also known as **nullity**. It is denoted by $\mu(G)$.

(ie) $\mu(G) = e - n + k$

UNIT - III

INCIDENT MATRIX

Matrix representation of graphs is a technique used to represent graph structures as numerical matrices (commonly adjacency or incidence matrices), enabling the application of linear algebra to analyse graph properties. These matrices store connections between vertices (nodes) and edges in tabular form, making them suitable for computer algorithms, with common types including Adjacency Matrix, Incidence Matrix, and Path Matrix.

INCIDENCE MATRIX:

Let G be a graph with n vertices e edges and no self-loops. Define an n by e matrix $A = [a_{ij}]$, whose nj rows correspond to the n vertices and the e columns correspond to the e edges, as follows

$$a_{ij} = \begin{cases} 1 & , if\ j^{th}\ edge\ e_j\ is\ incident\ on\ i^{th} vertex\ v_i \\ 0 & , otherwise \end{cases}$$

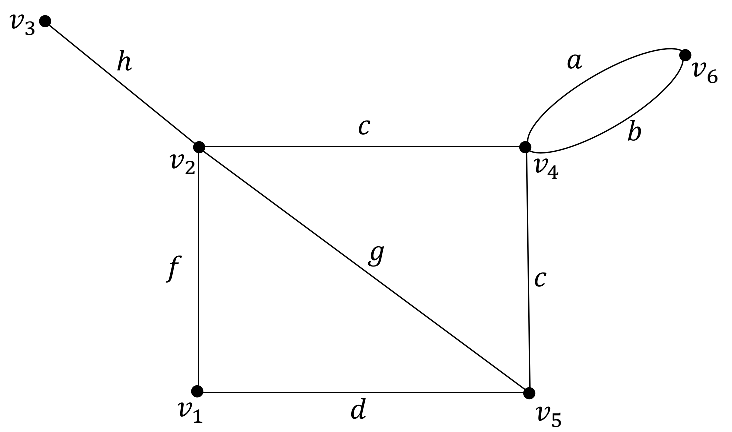


Fig: 3.1

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
A(G) = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

Such a matrix A is called the vertex-edge incidence matrix. Matrix A for a graph G is sometimes also written as $A(G)$.

The incident matrix contains only two elements 0 and 1. Such a matrix is called a binary matrix or a (0,1)-matrix.

OBSERVATIONS:

1. Since every edge is incident on exactly two vertices each column of A has exactly two $1's$.
2. The number of $1's$ in each row equals the degree of the corresponding vertex.
3. A row with all $0's$, therefore represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example columns 1 and 2 in Fig 3.2
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $A(G)$ of graph G can be written in a block-diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

where $A(g_1)$ and $A(g_2)$ are the incidence matrices of component5s g_1 and g_2. The observation results from the fact that no edge in g_1 is incident on vertices of g_2 and vice versa. Obviously, this remark is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabelling the vertices and edges of the same graph.

Theorem 3.1:

Two graph G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutations of rows and columns.

Proof:

Let G_1 and G_2 be two isomorphic graphs. Then there is a one to one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved.

Hence $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

Conversely, Assume that incidence matrix $A(G_1)$ and $A(G_2)$ differ only by permutation are rows and columns.

This corresponds to the relabelling the vertices and edges of the same graph.

Therefore, G_1 and G_2 are isomorphic.

Hence Proved.

Rank of the Incident Matrix:

If $A(G)$ is an incidence Matrix of a connected graph with n vertices, then $\text{rank}\big(A(G)\big) = n - 1$.

Theorem 3.2:

If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G)$ is $n-1$.

Proof:

Let G be a connected graph with n vertices and e be the number of edges in G.

Let $A(G)$ be the incidence matrix.

Each row in $A(G)$ is a binary vector over a vector space with addition modulo 2.

Let $A_1, A_2, \dots, A_n$ be the row vector of $A(G)$. Then $(G) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \dots \\ A_n \end{bmatrix}$, since there two one's in every column of $A(G)$.

Then $A_1 + A_2 + \cdots + A_n = [2 + 2 + \cdots + 2] \equiv 0(mod\ 2)$

We have $A_1, A_2, \dots, A_n$ are linearly dependent.

Hence, rank $\big(A(G)\big) \leq n-1$. ------------ (1)

Consider the sum of any m of the row vectors of $A(G)$, $m \leq n-1$.

Since G is connected, $A(G)$ cannot be partitioned in the form

$$A(G) = \left[\begin{array}{c|c} A(G_1) & 0 \\ \hline 0 & A(G_2) \end{array}\right]$$

Such that $A(G_1)$ has m rows and $A(G_2)$ has $n - m$ rows.

There exists no $m \times m$ submatrix of $A(G)$ for $m \leq n - 1$. Such that, the modulo 2 sum of these m rows is equal to zero. As there are only two elements 0 and 1 in the field, the addition of all vectors taken m at a time for $m = 1, 2, \dots, n - 1$ gives all possible linear combination of $n - 1$ row vector.

Hence, no linear combinations of m row vectors of A for $m \leq n - 1$ is zero.

Therefore, rank $\left(A(G)\right) \geq n - 1$ ---------------- (2)

From (1) and (2), we get

rank $\left(A(G)\right) = n - 1$.

Hence Proved.

Corollary 3.1:

The reduced incidence matrix of a tree is nonsingular.

Proof:

Let G be a connected graph.

A tree with n vertices has $n - 1$ edges and is also connected.

So the incidence matrix has n rows and $n - 1$ columns.

Since the graph is connected, the rank of the incidence matrix is $n - 1$..

Therefore, the reduced matrix is a square matrix of order $(n - 1) \times (n - 1)$ with rank $n - 1$.

So the reduced incidence matrix is non-singular.

Hence Proved.

CIRCUIT MATRIX:

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a circuit matrix $B = [b_{ij}]$ of G is a q by e, $(0,1)$ matrix defined as follows

$$b_{ij} = \begin{cases} 1 & , if\ i^{th}\ circuit\ includes\ j^{th}\ edge \\ 0 & , otherwise \end{cases}$$

The graph in Fig 3.1 has four different circuits, $\{a,b\}, \{c,e,g\}, \{d,f,g\}$ and $\{c,d,f,e\}$. Therefore, its circuit matrix is a 4 by 8.

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

OBSERVATIONS:

A circuit matrix $B(G)$ of a graph G is follows:

1. A column of all zeros corresponds to a non-circuit edge (ie., an edge that does not belong to any circuit).
2. Each row of $B(G)$ is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self loop the corresponding row will have a single 1.
4. The number of $1's$ in a row is equal to the number of edges in the corresponding circuit.

5. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right]$$

where $B(g_1)$ and $B(g_2)$ are the circuit matrices of g_1 and g_2. This observation results from the fact that circuits in g_1 have no edges belonging to g_2 and vice cersa.
6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabelling the circuits and edges.
7. Two graph G_1 and G_2 will have the same circuit matrix if and only if G_1 and G_2 are 2-isomorphic.

Theorem 3.3:

Let B and A respectively, the circuit matrix and the incidence matrix (of a self-loop free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A that is $A.B^T = B.A^T = 0(mod\ 2)$. Where superscript T denotes the transposed matrix.

Proof:

Consider a vertex v and a circuit c in a graph G. Either v is in circuit c it is not.

If v is not in circuit c, there is no edge in the circuit that is incident on v.

On the other hand, if v is in the circuit the number of edges in the circuit that are incident on v is exactly 2.

Consider the i^{th} row in A and j^{th} row in B. Here the edges are arranged in the same order, so the non-zero entry in the corresponding same positions occurs in both rows only if the particular edge is incident on the i^{th} vertex and also in the j^{th} circuit.

If the i^{th}vertex is not in the j^{th} circuit then there is no such non-zero entry and the dot product of the 2 rows is 0.

If the i^{th} vertex is in the j^{th} circuit there will be exactly two $1's$ in the sum of the product of two entries and the dot product of the 2 arbitrary rows, one from A and other from B is 0.

Therefore, $A.B^T = B.A^T = 0$

Hence Proved.

FUNDAMENTAL CIRCUIT MATRIX AND RANK OF *B*:

A submatrix of a circuit in which all rows correspond to a set of fundamental circuits is called a fundamental circuit matrix. It is denoted by B_f.

A graph and its fundamental circuit matrix with respect to a spanning tree.

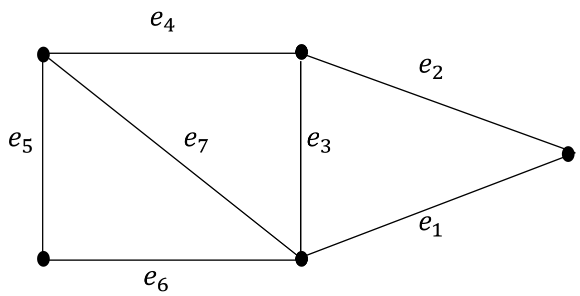


Fig: 3.2 (a)

Spanning Tree:

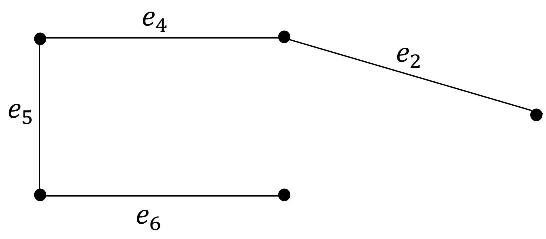


Fig: 3.2 (b)

In Fig: 3.2 (b), Branches $= \{e_2, e_4, e_5, e_6\}$ and Chords $= \{e_1, e_3, e_7\}$

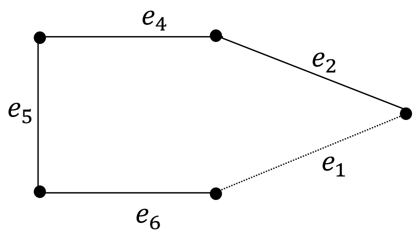


Fig: 3.2 (c)

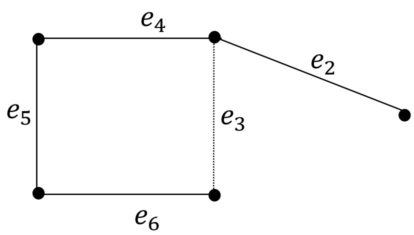


Fig: 3.2 (d)

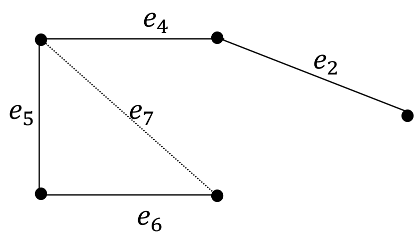


Fig: 3.2 (e)

These are the Fundamental circuit, in Fig: 3.2 (c) is $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in Fig: 3.2 (d) is $\{e_3, e_4, e_5, e_6\}$ and in Fig 3.2 (e) is $\{e_5, e_6, e_7\}$.

Fundamental circuit matrix is

$$B_f = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

If n is the number of vertices and e is the number of edges in a connected graph, then B_f is an $(e-n+1) \times c$ matrix, where $(e-n+1)$ is the number of chords in the spanning three, which corresponds to the number of fundamental circuits as each fundamental circuit is being produced by adding 1 chord.

Let us arrange the row as the first row corresponds to the fundamental circuit made by the chord in the $1st$ column and the second row to the fundamental circuit made by the second chord in the $2nd$ column and so on.

$$B_f = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc:cccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccc:cccc} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

$\Longrightarrow A$ matrix B_f thus arranged can be written as $B_f, [I_\mu \;\; B_t]$,

where, I_μ is the identity matrix of order μ. (ie) $\mu = e - n + 1$ (no. of chords)

B_t is the remaining submatrix corresponding to the branches of the spanning tree.

It is clear that rank of $B_f = e - n + 1 = \mu$.

Theorem 3.4:

If B is a circuit matrix of a connected graph G with e edges and n vertices, rank of $B = e - n + 1$.

Proof:

Let A be a incidence matrix and B be a circuit matrix. Let n be the number of vertices and e be the number of edges.

Since A and B are orthogonal, so $A.B^T = B.A^T = 0(mod\ 2)$

We know that rank of $A = n - 1$

According to sylvester's theorem

Rank of A + Rank of $B \leq e$

Rank of $B \leq e -$ Rank of A

$\leq e - (n-1)$

$\leq e - n + 1$ ---------------- (1)

Also, rank of $B \geq e - n + 1$ ---------- (2)

From (1) and (2) we have

rank of $B = e - n + 1.$

Hence Proved.

PATH MATRIX:

Let G be a graph with m edges, u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u.v) = [P_{ij}]_{q \times m}$, where q is the number of different paths between u and v is defined as

$$P_{ij} = \begin{cases} 1\ , if\ j^{th}\ edge\ lies\ in\ the\ i^{th} path \\ 0\ , otherwise \end{cases}$$

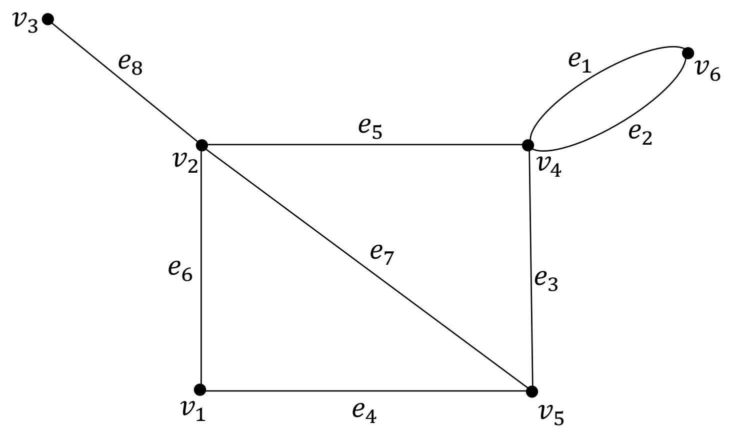


Fig: 3.3

There are the different paths between v_3 and v_4.

$P_1 = \{e_8, e_5\}$

$P_2 = \{e_8, e_7, e_3\}$

$P_3 = \{e_8, e_6, e_4, e_3\}$

Therefore, the path matrix for the vertices v_3 and v_4 is

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

OBSERVATIONS:

1. A column of all zero's corresponds to an edge that does not lie in any path between x and y.
2. A column od all one's corresponds to an edge that lie in every path between x and y.
3. There is no row with all zero's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

Theorem 3.5:

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$ of a connected graph are arranged in the same order, then the product $(mod\ 2)$, $A.P^T(x, y) = M$, where the matrix M has $1's$ in two rows x and y and the rest pf the $n - 2$ rows are all $0's$.

Proof:

Let G be connected graph and let $v_k = x$ and $v_t = y$ be any two vertices of G. Let A be the incidence matrix and $P(x, y)$ be path matrix (x, y) in G.

For any vertex v_i in G and for any xy path P_j in G, either $v_i \in P_j$ or $v_i \notin P_j$.

If $v_i \notin P_j$, then there is no edge of P_j which is incident on v_i.

If $v_i \in P_j$, then either v_i is an intermediate vertex of P_j or $v_i = v_k$ or $v_i = v_t$.

In case v_i is an intermediate vertex of P_j, then there are exactly two edges of P_j which are incident on v_i.

In case $v_i = v_k$ or $v_i = v_t$, there is exactly one edge of P_j, which is incident on v_i.

Consider the i^{th} row of A and j^{th} row of P, then $[AP^T(x, y)]_{ij}$.

As the edges are arranged in the same order, the r^{th} entries of these rows are both non-zero iff the edge e_r is incident on the i^{th} vertex v_i and also on the j^{th} path P_j.

Let $[AP^T(x, y)]_{ij} = [m_{ij}]$

we have $[AP^T]_{ij} = \sum_{r=1}^{m} [A]_{ir} [P^T]_{rj}$

Therefore, $m_{ij} = \sum_{r=1}^{m} a_{ir}. P_{jr}$

For each edge e_r of G, we have the following cases:

Case 1: e_r is incident on v_i and $e_r \notin P_j$.

Here $a_{ir} = 1$, $b_{jr} = 0$

Case 2: e_r is not incident on v_i and $e_r \in P_j$

Here $a_{ir} = 0$, $b_{jr} = 1$

Case 3: e_r is not incident on v_i and $e_r \notin P_j$

Here $a_{ir} = 0$, $b_{jr} = 0$

In all those cases, $m_{ij} = 0$

Case 4: e_r is incident on v_i and $e_r \in P_j$

If v_i is an intermediate vertex of P_j, then there are exactly two edges, say e_r and e_t incident on v_i.

So, $a_{ir} = 1$, $a_{it} = 1$, $P_{jr} = 1$ and $P_{jt} = 1$.

Therefore, $m_{ij} = 1 + 1 = 0(mod\ 2)$

If $v_i = v_k$ or v_t, then the edge e_r is incident on either v_k or v_t.

So, $a_{kr} = 1$, $P_{jr} = 1$ or $a_{tr} = 1$, $P_{jr} = 1$

Thus, $m_{kj} = \sum a_{kr}.P_{jr} = 1.1 = 1(mod\ 2)$

or

$m_{tj} = \sum a_{tr}.P_{jr} = 1.1 = 1(mod\ 2)$

In either case. $m_{ij} = 1(mod\ 2)$. Hence $m = [m_{ij}]$ is a matrix such that under modulo 2.

$$m_{ij} = \begin{cases} 1, for\ i = k, t \\ 0, otherwise \end{cases}.$$

Hence proved.

ADJACENCY MATRIX:

Let G ne a simple graph with n vertices. Its adjacency matrix is denoted by $X = [x_{ij}]$ and defined by $x_{ij} = \begin{cases} 1, \text{if there exists an edge between } v_i \text{ and } v_j \\ 0, \text{otherwise} \end{cases}$

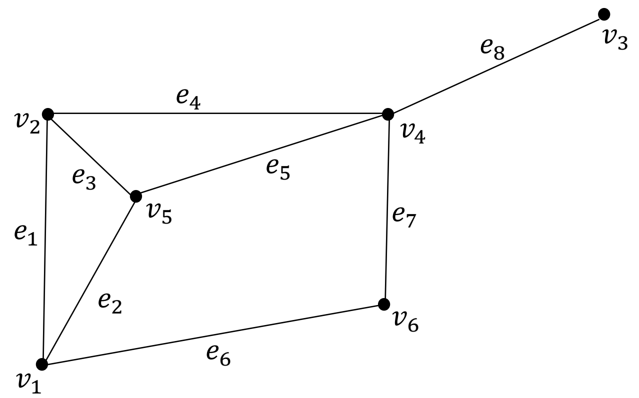


Fig: 3.4

$$X = [x_{ij}] = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

OBSERVATIONS:

It can made immediately about the adjacency matrix X of a graph G are:

1. The entries along the principal diagonal of X are all $1's$ if and only if the graph has no self-loops. A self-loop at the i^{th} vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.
3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of $1's$ in the corresponding row or column of X.
4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graph G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related:

   $$X(G_2) = R^{-1}.X(G_1).R$$

   where R is a permutation matrix.

5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right]$$

where $X(g_1)$ is the adjacency matrix of the component g_1 and $X(g_2)$ is that of the component g_2.

This partitioning clearly implies that there exists no edge joining any vertex in subgraph g_1 to any vertex in subgraph g_2.

6. Given any square, symmetric, binary matrix O of order n, one can always construct a graph G of n vertices (and no parallel edges) such that O is the adjacency matrix of G.

POWERS OF X:

Let us multiply by itself the 6 by 6 adjacency matrix of the simple graph in Fig: 3.4. The result, another 6 by 6 symmetric matrix X^2 is shown below.

$$X^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

The value of an off-diagonal entry in X^2, that is ij^{th} entry $(i \neq j)$ in X^2,

- number of $1's$ in the dot product of i^{th} row and j^{th} column (or j^{th} row) of X.
- number of positions in which both i^{th} and j^{th} rows of X have $1's$.
- number of vertices that are adjacent to both i^{th} and j^{th} vertices.
- number of different paths of length two between i^{th} and j^{th} vertices.

Theorem 3.6:

Let X be the adjacency matrix of a simple graph G. Then the ij^{th} entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

Proof:

Case 1: $r = 1$

$$(X^1)_{ij} = x_{ij}$$

By definition of the adjacency matrix, $x_{ij} = \begin{cases} 1 \text{ , if there is an edge from } v_i \text{ to } v_j \\ 0, \text{otherwise} \end{cases}$ So $(X)_{ij}$ counts the number of walks of length 1 from v_i to v_j.

Case 2: Inductive Hypothesis

Assume that for some $r \geq 1$,

$(X_r)_{ij}$ =number of walks of length r from v_i to v_j.

Inductive Step: Prove for $r + 1$.

Consider $X^{r+1} = X^r \cdot X$

The (i, j)-th entry is $(X^{r+1})_{ij} = \sum_{k=1}^{n} (X^r)_{ik} \cdot x_{kj}$

Now interpret each term:

- $(X^r)_{ik}$ = number of walks of length r from v_i to v_k (by hypothesis)
- x_{kj} = equals 1 if there is an edge from v_k to v_j, else 0.

Thus, $(X_r)_{ik} \cdot x_{kj}$ counts the number of walks of length $r + 1$ from v_i to v_j that pass through v_k as the second-to-last vertex. Summing over all k, we count all possible walks of length $r + 1$ from v_i to v_j.

Hence, $(X^{r+1})_{ij}$ = number of walks of length $r + 1$ from v_i to v_j is hold.

By mathematical induction, for all $r \geq 1$,

$(X^r)_{ij}$ = number of edge sequences (walks) of length r from v_i to v_j.

Hence Proved.

Corollary 3.2:

In a connected graph, the distance between two vertices v_i and v_j (for $i \neq j$) is k, if and only if k is the smallest integer for which the i, j^{th} entry in x^k is nonzero.

Corollary 3.3:

If X is the adjacency matrix of a graph G with n vertices and $Y = X + X^2 + x^3 + \cdots + x^{n-1}$ (in the ring of integers), then G is disconnected if and only if there exists at least one entry in matrix Y that is zero.

UNIT - IV

PLANAR GRAPH

Definition 4.1:

A **planar graph** is a graph that can be drawn on a flat surface (plane) such that no edges intersect each other, except at their endpoints (vertices).

In a planar graph:

- Edges only meet at vertices.
- The drawing divides the plane into regions called faces.

A key property of planar graphs is **Euler's Formula**:

$$V - E + F = 2$$

where V = vertices, E = edges and F = faces.

Example: A triangle or square is planar, while some graphs (like K_5) cannot be drawn without edge crossings and are **non-planar**.

KURATOWSKI'S TWO GRAPHS:

There are two types of Kuratowski's graph.

(i) Complete graph with five vertices. It is denoted by K_5.
(ii) Complete bipartite graph $K_{3,3}$.

K_5 is called first kuratowski's graph and $K_{3,3}$ is called second graph of Kuratowski's. K_5 is the non-planar graph with smallest number of vertices and $K_{3,3}$ is the non-planar graph with smallest number of edges.

Theorem 4.1:

The complete graph of five vertices is nonplanar.

Proof:

To prove this, we use Euler's Formula and a property of planar graph.

By Euler's Formula, For any connected planar graph $V - E + F = 2$.

Let V = number of vertices

E = number of edges

F = number of face (or) region

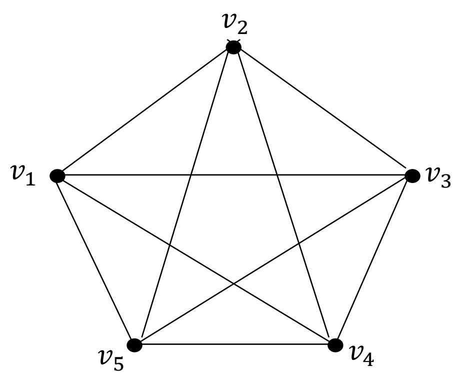


Fig 4.1

Here, the number of vertices $V = 5$ ------------ (1)

and the nuber of edges $E = \frac{5\times4}{2} = 10$ ---------- (2)

Now, $3V - 6 = 3(5) - 6 = 9$ ------------------- (3)

Here we see that, $10 \nless 9$

That implies, $E \nless 3V - 6$ ------------------------ (4)

We know that for a planar graph $E \leq 3V - 6$ --------- (5)

Comparing (4) and (5), contradict with each other

Therefore, K_5 is not a planar graph.

Hence Proved.

Theorem 4.2:

Kuratowski's second graph is also non-planar.

Proof:

By Euler's formula $E \leq 3V - 6$ but it is not planar.

Here, the vertices $V = 6$, and the edges $E = 9$ of the graph $K_{3,3}$.

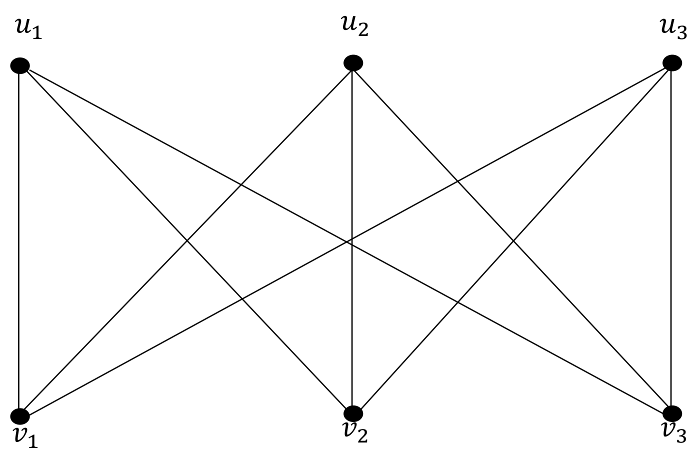


Fig 4.2

$$3V - 6 = 3(6) - 6 = 12$$

This relation satisfies the formula $E \leq 3V - 6$ but we shall show that $K_{3,3}$ is non-planar.

Let V = number of vertices

E = number of edges

F = number of face (or) region

Therefore, for $K_{3,3}$ of vertices $V = 6$ and edges $E = 9$

Since, in $K_{3,3}$ we see that each closed walk has at least four edges and degree of the region is the length of the closed walk.

Therefore, sum of the degree of region will be at least 4F. ----------- (1)

Since, number of edges E.

Total sum of the region $= 2E$ ---------------- (2)

From (1) and (2), we get

$4F \leq 2E \Rightarrow F \leq \frac{E}{2}$ ------------------------- (3)

For a planar graph, by Euler's formula

$V - E + F = 2$

$V - E + \frac{E}{2} \geq 2$

$V - \frac{E}{2} \geq 2$

$2V - E \geq 4$

$2(6) - 9 \geq 4$

$3 \ngeq 4$

which is contradiction

Therefore, $K_{3,3}$ is a non-planar graph.

Hence Proved.

DIFFERENT REPRESENTATIONS OF A PLANAR GRAPH:

Theorem 4.3:

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Proof:

We prove the theorem by **induction on the number of vertices**.

For a graph with $n \leq 3$ vertices, the result is trivial since such graphs can be drawn in the plane with straight-line edges without crossings.

Assume that every simple planar graph with fewer than nnn vertices can be drawn in the plane with all edges as straight-line segments.

Let G be a simple planar graph with n vertices.

From planar graph properties, G has at least one vertex v of degree ≤ 5.

Remove vertex v and all edges incident to it. Let the resulting graph be G'.

- G' is still a simple planar graph

- It has $n-1$ vertices

By the induction hypothesis, G' can be drawn in the plane such that all edges are straight-line segments and no edges intersect.

Now add the vertex v back into the drawing.

Let its neighbors be $v_1, v_2, \dots, v_k$, where $k \leq 5$. These vertices lie on the boundary of some face in the drawing of G'

Place v inside that face and join it to each v_i using straight-line segments.

Since, $k \leq 5$ and the neighbours surround a region (face),

we can place v such that the new edges do not cross any existing edges.

Thus, we obtain a straight-line drawing of G.

By induction, the theorem holds for all simple planar graphs.

Hence, every simple planar graph can be embedded in the plane so that all edges are straight-line segments.

Hence Proved.

Region:

The plane representation of a graph divides the plane into region (also called windows, faces or meshes), as shown in Fig 4.3.

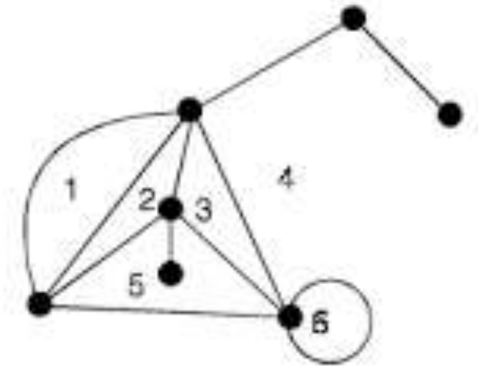


Fig 4.4

Infinite Region:

The portion of the plane lying outside a graph embedded in a plane, such as region 4 in Fig 4.4, is infinite in its extent. Such a region is called the infinite, unbounded, outer or exterior region for the particular plane.

Embedding on a sphere:

To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole).

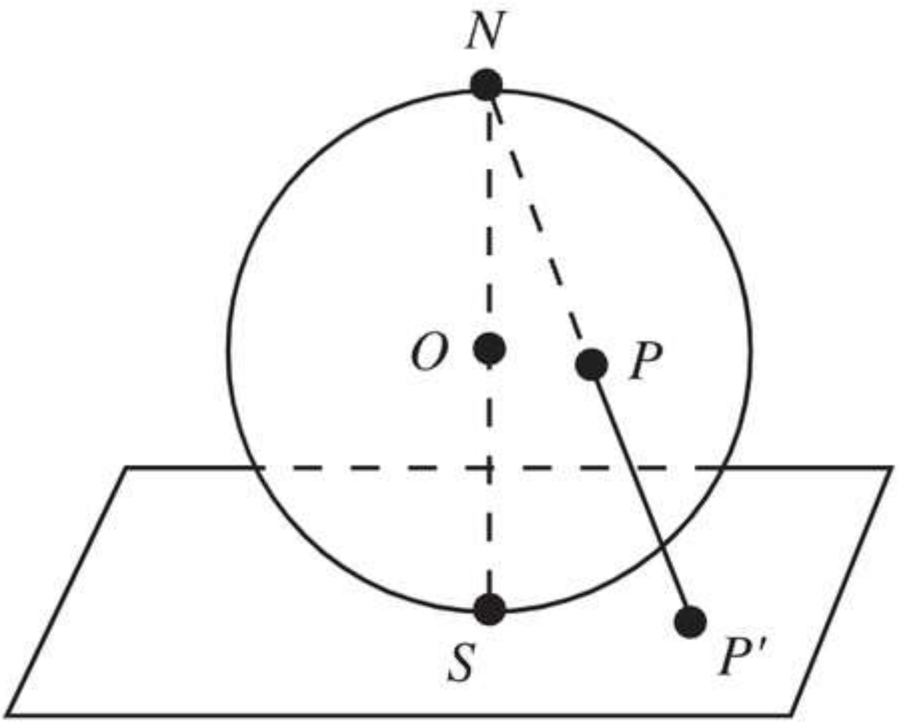


Fig 4.5

Now, corresponding to any point p on the plane, there exists a unique point p' on the sphere and vice versa, where p' is the point at which the straight line from point p to point NP intersects the surface of the sphere. Thus, there is a one-to-one correspondence between the points of the sphere and the finite points on the plane and points at infinity in the plane correspond to the point NP on the sphere.

From this construction, it is clear that any graph that can be embedded in a plane (ie draw on a plane such that its edges do not intersect) can also be embedded in the surface of the sphere and vice versa.

Theorem 4.4:

A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

Proof:

Case 1: Planar $\Longrightarrow$ Sphere

Assume a graph G is planar, meaning it can be drawn in a plane R^2 without edge intersections.

Consider a sphere S sitting on this plane touching it at a south pole SP.

Let N be the north pole the point on the surface furthest from SP. For any point p in the plane drawing of G, draw on a straight line from N to p. This line intersects the sphere at exactly one point p'. This maps the plane drawing of the graph onto the sphere surface. The only point on the sphere not covered is N, which can be chosen to lie in a face of the graph (not on an edge or vertex). Thus, the graph is embedded on the sphere.

Case 2: Sphere $\Longrightarrow$ Planar

Assume a graph G is embedded on a sphere S without edges intersecting.

Choose any point N on the sphere that is not on an edge or vertex of the graph.

Let a plane P be tangent to the sphere at the opposite point south pole SP.

Perform a stereographic projection for every point x on the sphere $(x \neq N)$, draw a l;ine passing through N and x until it intersects the plane P at x'.

This projection is a continuous, one-to-one mapping that transforms the sphere embedding into a plane drawing without edges intersecting.

Hence Proved.

Theorem 4.5:

A planar graph may be embedded in a plane such that any specified region (ie specified by the edges forming it) can be made the infinite region.

Proof:

A planar graph embedded on the surface of the sphere divides the surface of the sphere into different regions. Each region of the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP. Clearly, by suitably rotating the sphere, we can make any specified region map onto the infinite region on the plane.

Hence Proved.

Theorem 4.6:

In a connected planar graph with n vertices, e edges has $e - n + 2$ regions.

Proof:

Without loss of generality, assume that the planar graph is simple. Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygon net). Let the polygon net representing the given graph consist of f regions. Let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3k_3 + 4k_4 + 5k_5 + \cdots + rk_r = 2e \qquad \text{-------------- (1)}$$

where k_r is the number of polygons with r edges.

Also, $k_3 + k_4 + \cdots + k_r = f$ --------------- (2)

The sum of all angles subtended at each vertex in the polygon net in $2\pi n$.

------- (3)

Now, the sum of all interior angles of a p-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$. The expression in (3) is the total sum of all interior angles of $f-1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\pi(3-2)k_3 + \pi(4-2)k_4 + \ldots + \pi(r-2)k_r + 4\pi$$

$$= \pi[3k_3 + 4k_4 + \ldots + rk_r - 2(k_3 + k_4 + \ldots + k_r)] + 4\pi$$

$$= \pi(2e - 2f) + 4\pi = 2\pi(e - f + 2) \qquad \text{------------------ (4)}$$

Equating (3) and (4) we get

$$2\pi(e - f + 2) = 2n\pi, \text{ so that } f = e - n - 2.$$

Hence Proved.

Corollary 4.1:

In any simple, connected planar graph with f regions, n vertices and e edges $(e > 2)$, the following inequalities must hold: $e \geq \frac{3}{2}f$ or $e \leq 3n - 6$.

Proof:

First assume that the planar graph G is connected. If $n = 3$, then since G is simple, therefore G has at most three edges. Thus $m \leq 3$, that is, $m \leq 3 \times 3 - 6$, and the result is true.

Now, let $n = 4$. First let G be a tree so that $m = n - 1$. Since $n \geq 4$, obviously we have $n - 1 \leq 3n - 6$. Now, let G be not a tree. Then G has cycles. Clearly, there is a cycle in G, all of whose edges lie on the boundary of the exterior region of G. Then, since G is simple, we have $d(\varphi) \geq 3$, for each region φ of G. Let $b = \sum_{\varphi \in F} d(\varphi)$, where F denotes the set of all regions of G.

Since each region has at least three edges on its boundary, therefore we have $b \geq 3f$, where f is the number of regions of G. However, when we sum to get b, each edge of G is counted either once or twice (twice when it occurs as a boundary edge for two regions). Therefore, $b \leq 2e$ so that $3f \leq b \leq 2e$. In particular, we have $3f \leq 2e$. Using Euler's formula $n - e + f = 2$, we get $e \leq 3n - 6$. Let G be not connected and let $G_1, G_2, \ldots, G_t$ be its connected components. For each i, $1 \leq i \leq t$, let n_i and e_i denote the number of vertices and edges in G_i. Since each G_i is a planar simple graph, by the above argument, we have $e_i \leq 3n_i - 6$, for each i, $1 \leq i \leq t$.

Also, $n = \sum_{i=1}^{t} n_i$ and $m = \sum_{i=1}^{t} e_i$.

Hence, $m = \sum_{i=1}^{t} e_i \leq \sum_{i=1}^{t}(3n_i - 6) = 3\sum_{i=1}^{t} n_i - 6t \leq 3n - 6$.

Hence Proved.

Theorem 4.7:

The spherical embedding of every planar 3-connected graph is unique.

Proof:

Let G be a 3-connected planar graph. Suppose it has two different embeddings. In any embedding, the edges around each vertex have a cyclic order.

Because G is 3-connected, there exist at least three internally disjoint paths between vertices. This strong connectivity fixes the cyclic order of edges around each vertex. Any change in this order would create edge crossings, violating planarity.

Also, the faces of the graph are determined by cycles, which remain the same in all embedding's. Hence, no different embedding is possible.

Therefore, the embedding of a 3-connected planar graph is unique up to homeomorphism.

Hence Proved.

COLORING, COVERING AND PARTITIONING:

In Graph Theory, the concepts of coloring, covering, and partitioning are fundamental techniques used to study the structure and properties of graphs. These ideas are widely applied in scheduling, network design, optimization, and computer science.

1. Graph Coloring

Graph coloring involves assigning colors to elements of a graph under certain constraints.

- **Vertex Coloring:** Assign colors to vertices such that no two adjacent vertices share the same color.
- **Edge Coloring:** Assign colors to edges so that no two edges sharing a common vertex have the same color.
- The minimum number of colors required is called the **chromatic number**. It is denoted by $\chi(G)$.

Applications: Timetable scheduling, map coloring, frequency assignment.

2. Graph Covering

A covering in a graph refers to selecting a set of elements that “cover” all parts of the graph.

- **Vertex Cover:** A set of vertices such that every edge has at least one endpoint in the set.
- **Edge Cover:** A set of edges such that every vertex is incident to at least one selected edge.

Applications: Network monitoring, resource placement, security systems.

3. Graph Partitioning

Partitioning divides the graph into disjoint subsets that satisfy certain properties.

- **Vertex Partition:** Dividing vertices into groups such that each vertex belongs to exactly one group.
- **Edge Partition:** Dividing edges into disjoint sets.
- Often used to break a graph into smaller, manageable components.

Applications: Parallel computing, clustering, circuit design.

Coloring, covering, and partitioning provide powerful methods to analyze graphs by organizing their elements into structured forms. These concepts help solve real-world problems efficiently by simplifying complex relationships within networks.

CHROMATIC NUMBER:

The chromatic number of a graph G, denoted by $\chi(G)$ is the minimum number of colors required to color the vertices of the graph so that no two adjacent vertices share the same color. This represents a proper vertex coloring.

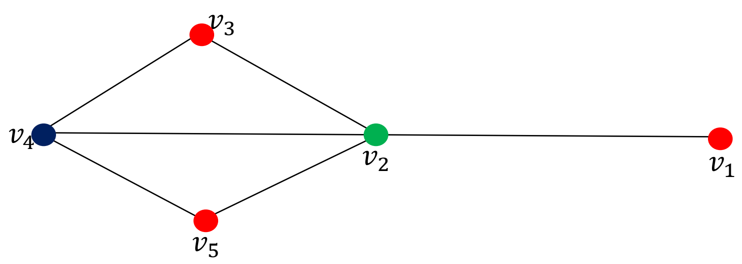


Fig 4.6

The proper coloring which is of interest to us is one that requires the minimum number of colors. A graph G that requires k different colors for its proper colouring and no less is called a k-chromatic graph and the number k is called the chromatic number of G. In Fig 4.6, the graph is 3-chromatic.

OBSERVATIONS:

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges (not a self-loop, of course) is at least 2-chromatic (also called bichromatic).
3. A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is at least r-chromatic. For instance, every graph having a triangle is at least 3-chromatic.
4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Theorem 4.8:

Every tree with two or more vertices is 2-chromatic.

Proof:

Let T be a tree with two or more vertices.

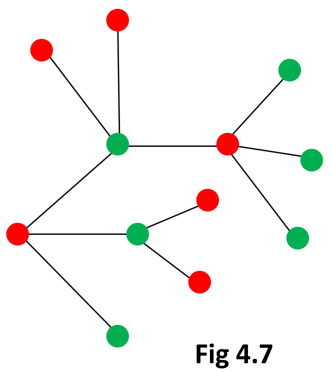


Fig 4.7

We need to consider any vertex n of T and let us assume T is rooted at vertex n.

First, we need to assign a color code of 1 to n.

Let us assign a color code of 2 to all vertices adjacent to n.

Let $n_1, n_2, \ldots, x$ be the vertices assigned with color 2.

Those which are adjacent to $n_1, n_2, \ldots, x$ are assigned with color 1.

We need to continue this process till every vertex in T has been assigned the color.

We observe that in T, vertices with odd distances from n will have a color code of 2, and those which are at even distances from n will have a color code of 1.

Therefore while going through any path in T, the vertices will have and show alternating colors.

Due to the presence of only one root and one path between two vertices within a tree, none of the two adjacent vertices will have the same color code.

Thus T will be colored in two colors.

Therefore, T is 2-chromatic.

Hence Proved.

Theorem 4.9:

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof:

Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G.

Let us properly color T with two colors.

Since G has all circuits of even length, the spanning tree is of odd length. The spanning tree can be coloured using 2 colors according to theorem 4.8. The first vertex and last vertex of this spanning tree will have different colors.

Now add the chords to T. Because the first and last vertex have different colors, when we add the chord and form the circuit, adjacent vertices will not have same color.

Thus the theorem. Conversely, if G has a circuit of odd length, we would need at least three colors.

Hence Proved.

Theorem 4.10:

If d_{max} is the maximum degree of the vertices in a graph G, chromatic number of $G \leq 1 + d_{max}$.

Proof:

Consider a graph G with maximum degree d_{max}.

Let the vertices be ordered as $v_1, v_2, v_3, \dots, v_n$. We color each vertex in order using at most $d_{max} + 1$ colors.

Now we assign the starting vertex v_1 choose color 1 and move to vertices $v_2, v_3, \dots$ so on.

When coloring any vertex v_i it has at most d_{max} adjacent vertices. So, at most d_{max} different colours are already used by its neighbours.

Since we have $d_{max} + 1$ colors available and at most d_{max} are used by neighbours at least one color always available. So we can always assign a valid color to v_i.

Thus, all vertices can be colored using at most $d_{max} + 1$ colors.

Therefore, $\chi(G) \leq d_{max} + 1$.

Hence Proved.

CHROMATIC PARTITIONING:

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example Fig 4.6 produces the partitioning

$$\{v_1, v_3, v_5\}, \{v_2\} \text{ and } \{v_4\}$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set.

Formally, A set of vertices in a graph is said to be an independent set of vertices or simply an independent set if no two vertices in the set are adjacent.

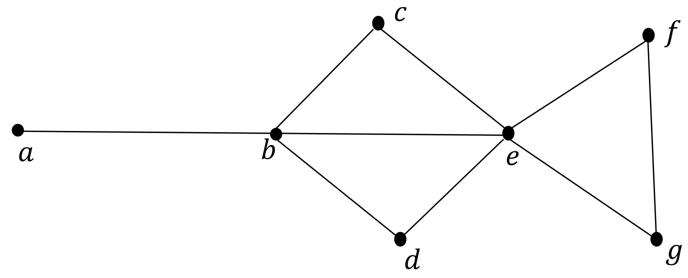


Fig 4.8

In Fig 4.8, $\{a, c, d\}$ is an independent set. A single vertex in any graph constitutes an independent set.

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property. The set $\{a, c, d, f\}$ is a maximal independent set. The set $\{b, f\}$ is another maximal independent set. The set $\{b, g\}$ is a third one.

The number of vertices in the largest independent set of a graph G is called the independent number $\beta(G)$.

Consider a k-chromatic graph G of n vertices properly colored with k different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number $\beta(G)$, we have the inequality $\beta(G) \geq \frac{n}{k}$.

Definition 4.2:

Given a simple, connected graph G, partiotion all vertices of G into the smallest possible number of disjoint, independent sets is known as the **chromatic partitioning** of graph.

For example, In Fig 4.8, four are some chromatic partitions of the graph.

$$\{(a, c, d, f), (b, g), (e)\},$$

$$\{(a, c, d, g), (b, f), (e)\},$$

$$\{(c, d, f), (b, g), (a, e)\},$$

$$\{(c, d, g), (b, f), (a, e)\}$$

Definition 4.3:

A graph that has only one chromatic partition is called a **uniquely colourable** graph. The graph in Fig 4.8 is not a uniquely colourable graph, but the one in Fig 4.9.

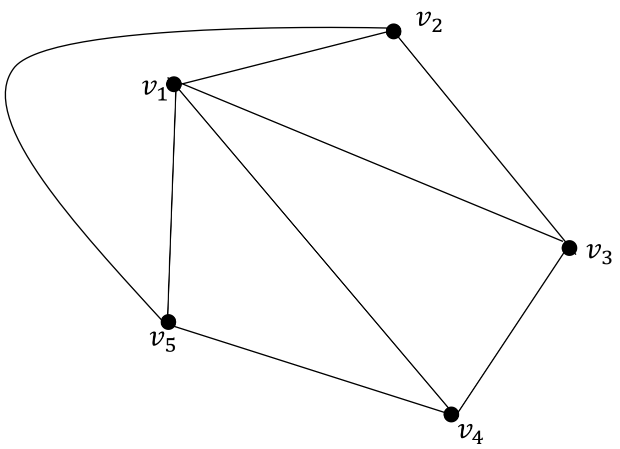


Fig 4.9

Definition 4.4:

A **dominating set** (or an externally stable set) in a graph G is a set of vertices that dominates every vertex v in G in the following sense:

Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set. For instance, the vertex set $\{b, g\}$ is a dominating set in Fig 4.8. So is the set $\{a, b, c, d, f\}$ a dominating set. A dominating set need not be independent. For example, the set of all its vertices is trivially a dominating set in every graph.

Definition 4.5:

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property. For example, in Fig 4.8, $\{b, e\}$ is a minimal dominating set. And so is $\{a, c, d, f\}$.

OBSERVATIONS:

1. Any one vertex in a complete graph constitutes a minimal dominating set.
2. Every dominating set contains at least one minimal dominating set.
3. A graph may have many minimal dominating sets, and of different sizes. [The number of vertices in the smallest minimal dominating set of a graph G is called the dominating number $\alpha(G)$].
4. A minimal dominating set may or may not be independent.
5. Every maximal independent set is a dominating set. For if an independent set does not dominate the graph, there is at least one vertex that is neither in the set nor adjacent to any vertex in the set. Such a vertex can be added to the independent set without destroying its independence. But then the independent set could not have been maximal.
6. An independent set has the dominance property only if it is maximal independent set. Thus an independent dominating set is the same as a maximal independent set.
7. In any graph G, $\alpha(G) \leq \beta(G)$.

Finding Minimal Dominating Set:

To dominate a vertex v_i we must either include v_i or any of the vertices adjacent to v_i. A minimum set satisfying this condition for every vertex v_i is a desired set. Therefore, for every vertex v_i in G let us form a Boolean product of sums $(v_i + v_{i1} + v_{i2} + \cdots + v_{in})$, where $v_{i1}, v_{i2}, \dots, v_{in}$ are the vertices adjacent to v_i and d is the degree of v_i

$$\theta = \Pi(v_i + v_{i1} + v_{i2} + \cdots + v_{in}) \text{ for all } v_i \text{in } G.$$

When θ is expressed as a sum of products, each term in it will represent a minimal dominating set.

CHROMATIC POLYNOMIAL:

In general, a given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows:

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in

$\binom{\lambda}{i}$ different ways, there are $c_i\binom{\lambda}{i}$ different ways of properly colouring G using exactly i colors out of λ colors.

Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}$$

Each ci has to be evaluated individually for the given graph. For example, any graph with even one edge requires at least two colors for proper coloring, and therefore $c_1 = 0$.

A graph with n vertices and using n different colors can be properly colored in $n!$ ways; that is, $c_n = n!$.

Theorem 4.11:

A graph of n vertices is a complete graph if and only if its chromatic polynomial is $\boldsymbol{P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}$.

Proof:

Necessary part:

Assume G is a complete graph K_n with n vertices. By definition, every pair of distinct vertices in K_n is connected by an edge. We determine the number of ways to properly color G using λ colors by considering the first vertices can be colored with any of the λ available colors.

The second vertices must be a different color from the first because they are adjacent. There are $\lambda - 1$ choices.

The third vertices must be different from both the first and second vertices (since it is adjacent to both). There are $\lambda - 2$ choices.

The k^{th} vertex is adjacent to all $k - 1$ previously colored vertices. Thus, it must have a color different from all of them, leaving $\lambda - (k - 1)$ choice.

For the final vertex of n^{th} vertex, there are $\lambda - (n - 1)$ choices. Multiplying these independent choice gives the chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$$

Sufficient part:

Assume a graph G with n vertices has the chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$

The root of this polynomial are exactly $0, 1, 2, \dots, n - 1$.

The chromatic number $\chi(G)$ is the smallest positive integer λ, such that $P_n(\lambda) > 0$. For this polynomial $P_{n-1}(\lambda) = 0$ and $P_n(\lambda) = n! > 0$.

Therefore, $\chi(G) = n$.

A fundamental property of the chromatic polynomial $P_n(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} + \cdots$ is that the absolute value of the coefficient of the second-highest power, $|a_{n-1}|$ is equal to the number of edges m in the graph.

Expanding $P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$ gives leading term of λ^n and a second term of $-\sum_{i=0}^{n-1} i\, \lambda^{n-1}$.

The sum $\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$. This means G must have exactly $\frac{n(n-1)}{2}$ edges.

In a simple graph with n vertices, the maximum possible number of edges is $\frac{n(n-1)}{2}$, which occurs only if the graph is complete. Since G has this exact number of edges, it must be K_n.

Hence Proved.

Theorem 4.12:

An n vertex graph is a tree if and only if its chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof:

To prove that an n-vertex graph G is a tree if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

We can prove this theorem by mathematical induction on the number of vertices n.

For $n = 1$, the graph is a single vertex. A tree with one vertex has a chromatic polynomial $P_n(\lambda) = \lambda$, which matches the formula $\lambda(\lambda - 1)^{1-1} = \lambda$.

Assume that for any tree T' with $n - 1$ vertices, its chromatic polynomial is $P_{T'}(\lambda) = \lambda(\lambda - 1)^{(n-1)-1} = \lambda(\lambda - 1)^{n-2}$.

Consider a tree T with n vertices. Every tree with $n \geq 2$ vertices has at least one vertex of degree 1. Let v be a leaf vertex connected to edge e and connected to the rest of the tree $T - v$.

To color the tree T, we first color the tree $T - v$ (which has $n - 1$ vertices) in $P_{T-v}(\lambda)$ ways. The vertex v is adjacent to only one vertex in $T - v$. Therefore,

v can be assigned any color except for the one color used by its neighbour, giving $(\lambda - 1)$ choices for vertex v.

Thus, the total number of colorings is $P_T(\lambda) = P_{T-v}(\lambda) \times (\lambda - 1)$.

Substituting the inductive hypothesis $P_T(\lambda) = \lambda(\lambda - 1)^{n-2} \times (\lambda - 1) = \lambda(\lambda - 1)^{n-1}$.

Next, we have to prove if $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$ then G is a tree.

A graph G is connected if and only if its chromatic polynomial $P_n(\lambda)$ is divisible by λ but not by λ^2. In the formula $\lambda(\lambda - 1)^{n-1}$, λ is a factor but $\lambda - 1$ is not a factor of the whole polynomial in a way that implies disconnection. Thus, G is connected.

The chromatic polynomial $P_n(\lambda)$ of a graph G with n vertices and m edges has the form $P_n(\lambda) = \lambda^n - m\lambda^{n-1} + \cdots$. Expanding $\lambda(\lambda - 1)^{n-1}$ using the binomial theorem gives $\lambda(\lambda^{n-1} - (n-1)\lambda^{n-2} + \cdots) = \lambda^n - (n-1)\lambda^{n-1} + \cdots$. Comparing the coefficients of λ^{n-1}, we find that $m = n - 1$.

Therefore, a connected graph with n vertices and $n - 1$ edges is a tree.

Hence Proved.

Theorem 4.13:

Let a and b two nonadjacent veretices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then $P_n(\lambda)$ of $G = P_n(\lambda)$ of $G' + P_{n-1}(\lambda)$ of G''.

Proof:

Let $P_n(\lambda)$ denote the number of **proper vertex colourings** of G using λ colours.

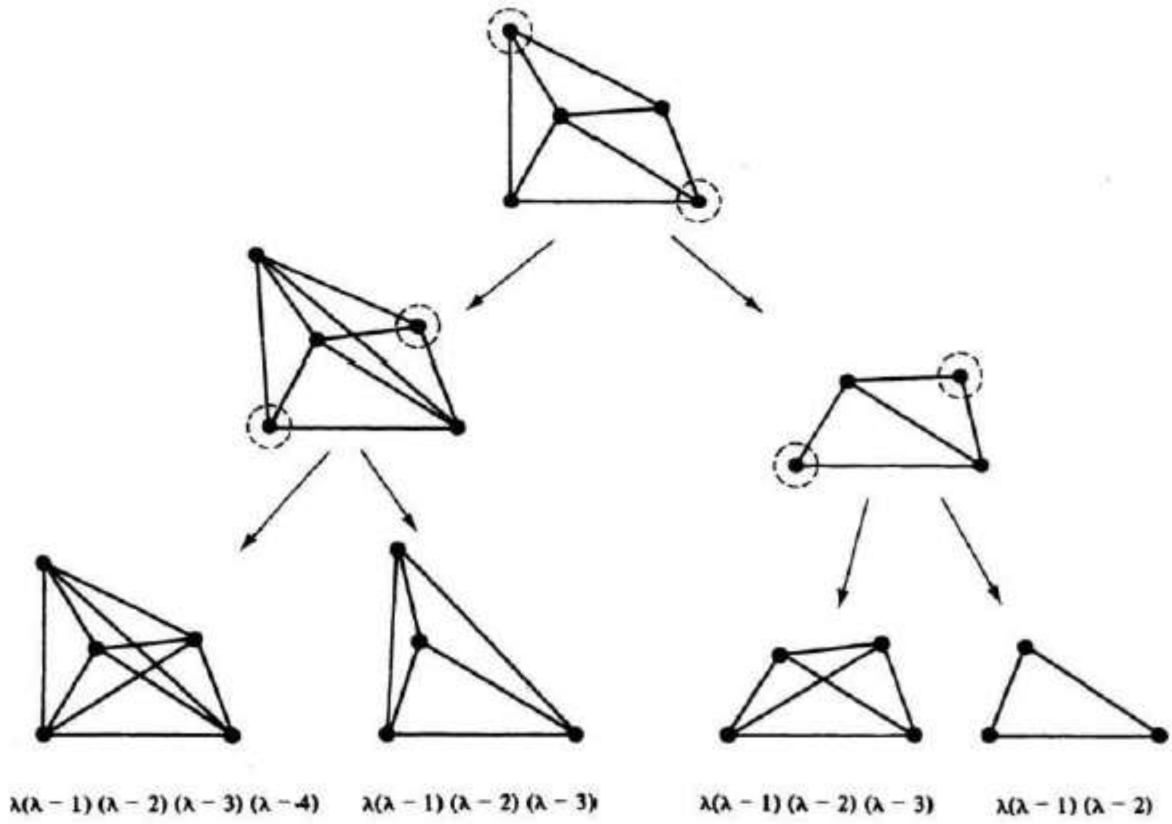


Fig 4.10

We classify all proper colourings of G into two mutually exclusive cases:

Case 1: Vertices a and b get different colours

Since a and b are assigned different colours, adding an edge between them does not violate the colouring condition.

Hence, every such colouring of G is also a proper colouring of G'.

Number of such colourings= $P_n(\lambda)$ of G'.

Case 2: Vertices a and b get the same colour

Since a and b have the same colour, we can treat them as a single vertex.

This is exactly the graph G'', obtained by fusing a and b.

Any proper colouring of G'' corresponds uniquely to such colourings of G.

Number of such colourings $= P_{n-1}(\lambda)$ of G''.

Combining both cases, every proper colouring of G falls into exactly one of the above cases. The two cases are disjoint and complete.

Therefore, $P_n(\lambda)$ of $G = P_n(\lambda)$ of $G' + P_{n-1}(\lambda)$ of G''.

Hence proved.

UNIT - V

MATCHINGS

In Graph Theory, a **matching** is one of the most important concepts used to study relationships between pairs of vertices in a graph.

Definition 5.1:

A **matching** in a graph $G = (V, E)$ is a set of edges such that no two edges share a common vertex**.** In other words, a matching selects edges so that each vertex is used at most once.

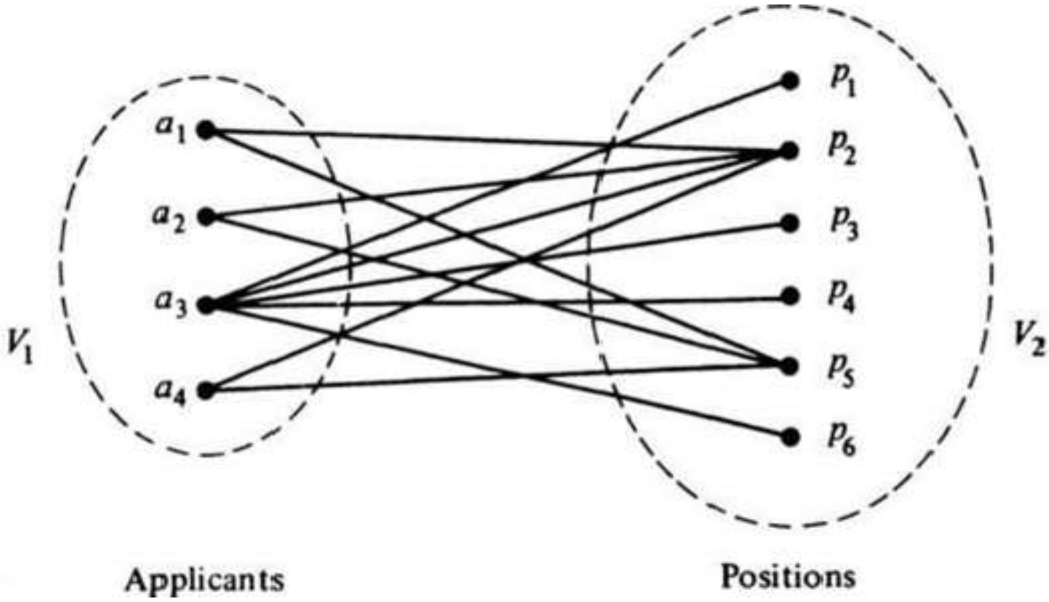


Fig 5.1

Definition 5.2: (Maximal Matching)

A matching that contains the largest possible number of edges. It is not necessarily unique. It is denoted by M, where $|M|$ is maximum.

(a) (b)

Fig 5.2

Definition 5.3: (Perfect Matching)

A matching in which every vertex of the graph is matched. Every vertex is incident with exactly one edge in the matching. Possible only when the number of vertices is even.

Example 5.4:

Suppose that four applicants a_1, a_2, a_3 and a_4 are available to fill six vacant positions p_1, p_2, p_3, p_4, p_5 and p_6. Applicant a1 is qualified to fill position p_2 or p_5. Applicant a_2 can fill p_2 or p_5. Applicant a_3 is qualified for p_1, p_2, p_3, p_4 or p_6. Applicant a_4 can fill jobs p_2 or p_5. This situation is represented by the graph in Fig.5.1. The vacant positions and applicants are represented by vertices. The edges represent the qualifications of each applicant for filling different positions. The graph clearly is bipartite, the vertices falling into two sets $V_1 = \{a_1, a_2, a_3, a_4\}$ and $V_2 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.

The questions one is most likely to ask in this situation are: Is it possible to hire all the applicants and assign each a position for which he is suitable? If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?

This is a problem of matching (or assignment) of one set of vertices into another. More formally, a matching in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is obviously a matching.

Theorem 5.1: (Hall's Marriage Theorem)

A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.

Proof:

Necessary part:

First, assume that a complete matching M of V_1 into V_2 exists.

By definition, every vertex $v \in V_1$ is matched to a distinct vertex $f(v) \in V_2$ via an edge in M.

Consider any subset $S \subseteq V_1$ with $|S| = r$ vertices. Each vertex in S is matched to a unique vertex in V_2. Because M is matching, these r matched vertices in V_2 are all distinct. All these r vertices are by definition, adjacent to vertices in S.

Therefore, the set of neighbours $N(S)$ must contain at least these r distinct vertices, so $|N(S) \geq r$. This hold for all r and all subsets S.

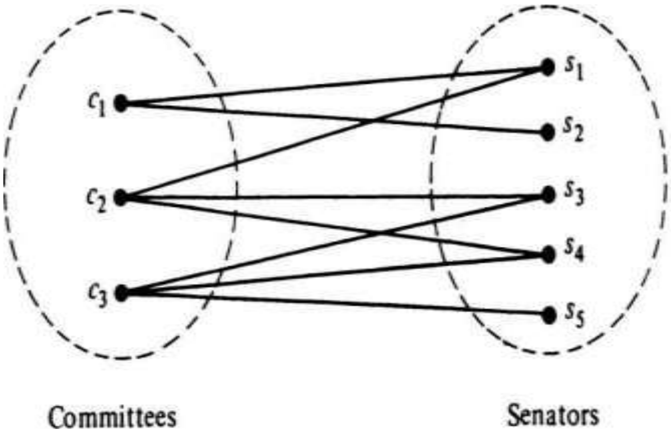


Fig: 5.3

Sufficient part:

We prove that if $|N(S)| \geq |S|$ for all $S \subseteq V_1$ a complete matching exists using strong induction on $n = |V_1|$.

If $n = 1$, V_1 has only one vertex $\{v\}$, then $|N(\{v\})| \geq 1$. This means v has at least one neighbour in V_2. Matching v to any neighbour gives a complete matching.

Assume the theorem hold for all bipartite graphs where the first partition has fewer than n vertices. Consider the two cases for a graph where $|V_1| = n$.

Case 1: $|N(S)| > |S|$ for every non-empty proper subset $S \subset V_1$.

Pick an arbitrary edge (u, v) where $u \in V_1, v \in V_2$. Match u to v and remove both from the graph. For any remaining subset $S' \subseteq V_1 - \{u\}$, its neighbourhood in the new graph $N'(S')$ has at least $|N'(S')| - 1$ vertices (since only v could have been removed from its original neighbourhood). Since $|N(S')| > |S'|$, we have $|N'(S')| \geq |S'|$. By induction hypothesis, a complete matching exists for the remaining $n - 1$ vertices.

Case 2: $|N(S)| = |S|$ for non-empty proper subset $S \subset V_1$.

Let S be a subset. By the inductive hypothesis, there is a complete matching for S into $N(S)$ because the condition hold for S. Now consider remaining vertices $V_1 - S$ and $V_2 - N(S)$. We must show that satisfy Hall's condition. For any $T \subseteq V_1 - S$, it neighbours in the remaining set $V_2 - N(S)$ must be at least $|T|$. If not, then $|N(S \cup T)| = |N(S) \cup N(T)| < |S| + |T|$, which contradicts the original condition $|N(S \cup T)| \geq |S \cup T|$. Thus, a matching exists for $V_1 - S$ into $V_2 - N(S)$ by induction. Combining the two matchings gives a complete matching for V_1.

Hence Proved.

Thoerem 5.2:

In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive integer m for which the following condition is satisfied degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

Proof:

Let S be any arbitrary subset of V_1. Let E_s be the set of all edges incident to the vertices in S. Since every vertex in V_1 has a degree of at least m, the total number of edges in E_s is $|E_S| \geq m.|S|$

All edges in E_S must, by definition terminate in the neighbourhood $N(S) \subseteq V_2$. Let $E_{N(S)}$ be the set of all edges incident to the vertices in $N(S)$. Since every vertex in V_2 has a degree of at most m, the total number of edges incident to $N(S)$ is

$$\left|E_{N(S)}\right| = m.|N(S)| \text{ ----------- } (1)$$

Because every edge in E_S is also an edge in $E_{N(S)}$ (since every edge from S must land in $N(S)$), we have the following inequality is

$$|E(S)| \leq |E_{N(S)}| \text{ --------------- } (2)$$

comparing the equation (1) and (2), we have

$$m.|S| \leq |E_S| \leq |E_{N(s)}| \leq m.|N(S)|$$

From the simplified inequality $m.|S| \leq m.|N(S)|$, we divide both sides by the positive integer m.

(ie) $|S| \leq |N(S)|$

Since this condition is holds for any arbitrary subsets $S \subseteq V_1$, Hall's condition is satisfied. Because Hall's condition $|N(S)| \geq |S|$ is satisfied for all $S \subseteq V_1$, there exists a matching in G that saturates every vertex in V_1. This is a complete matching in V_1 into V_2.

Hence proved.

Theorem 5.3:

The maximal number of vertices in set V_1 that can be matched into V_2 is equal to number of vertices in $V_1 - \delta(G)$.

Proof:

Let Mbe any matching in G, and suppose it matches k vertices of V_1. Then $| V_1 | -k$ vertices remain unmatched.

Consider any subset $S \subseteq V_1$. Since each vertex in $N(S)$ can match at most one vertex of S, number of matched vertices in $S \leq | N(S) |$.

Hence, at least $| S | - | N(S) |$ vertices in S must remain unmatched. Since this is true for all $\subseteq V_1$, number of unmatched vertices $\geq \delta(G)$.

Thus, $| V1 | - k \geq \delta(G)$

$k \leq | V1 | - \delta(G)$. ------------------- (1)

We now show that a matching exists that matches exactly $| V_1 | - \delta(G)$ vertices.

By **Hall's Theorem (generalized form)** a matching that leaves exactly $\delta(G)$ vertices unmatched exists.

That is, there exists a matching M such that

number of unmatched vertices in $V1 = \delta(G)$.

Thus, $k = | V1 | - \delta(G)$ --------------------- (2)

By theorem (1) and (2), we get

No matching can exceed $| V1 | - \delta(G)$, a matching achieving this bound exists.

Therefore, maximum number of vertices in V_1 that can be matched into $V_2 = | V_1 | - \delta(G)$.

Hence proved.

Matching and Adjacency Matrix:

Consider a bipartite graph G with nonadjacent sets of vertices V_1 and V_2, having number of vertices n_1 and n_2 respectively, and let $n_1 \le n_2, n_1 + n_2 = n$, the number of vertices in G. The adjacency matrix $X(G)$ of G can be written in the form

$$X(G) = \left[\begin{array}{c|c} 0 & X_{12} \\ \hline X_{12}^T & 0 \end{array}\right],$$

Where the submatrix X_{12} is the n_1 by n_2, $(0, 1)$-matrix containing the information as to which of the n_1 vertices of V_1 are connected to which of the n_2 vertices of V_2. Matrix is the transpose of X_{12}. Clearly, all the information about the bipartite graph G is contained in its X_{12} matrix. A matching V_1 into V_2 corresponds to a selection of the l's in the matrix X_{12} such that no line (i.e., a row or a column) has more than one 1. The matching is complete if the n_1 by n_2 matrix made of selected 1's has exactly one 1 in every row. For example, the X_{12} matrix for Fig: 5.3 is

$$X_{12} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$n_1 = 3, \quad n_2 = 5, \quad n = 8, \quad \text{and} \quad n_1 \le n_2,$$

$$V_1 = \{c_1, c_2, c_3\}$$

$$V_2 = \{s_1, s_2, s_3, s_4, s_5\}.$$

A complete matching of V_1 into V_2 is given by

$$\mathsf{M} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

A maximal matching corresponds to the selection of a largest possible number of 1's from X_{12} such that no row in it has more than one 1. Therefore, according to Theorem: 5.3, in matrix X_{12} the largest number of 1's, no two of which are in one row, is equal to number of vertices in $V_1 - \delta\,(G)$.

Definition 5.4: (Coverings)

In a graph G a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be an edge covering, a covering subgraph, or simply a covering of G.

For example, a graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. A Hamiltonian circuit (if it exists) in a graph is also a covering. We have already dealt with some coverings with specific properties, such as spanning trees and Hamiltonian circuits. In this section we shall investigate the *minimal covering* - a covering from which no edge can be removed without destroying its ability to cover the graph. In Fig:5.4 a graph and two of its minimal coverings are shown in heavy lines.

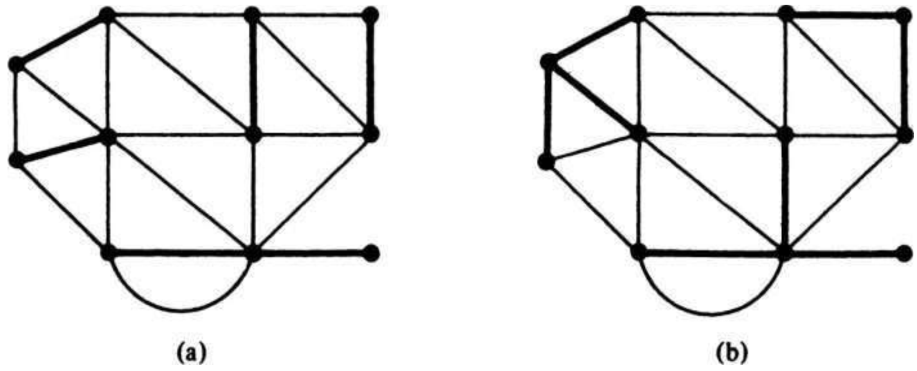


Fig: 5.4

OBSERVATIONS:

1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges. ($\lceil n \rceil$ denotes the smallest integer not less than x.)
3. Every pendant edge in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. If we denote the remaining edges of a graph by $(G - g)$, the set of edges g is a covering if and only if, for every vertex V, the degree of vertex in $(G - g) \leq$ (degree of vertex v in G) $- 1$.
6. No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than $n - 1$ edges.
7. In general, A graph has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number

of edges in a minimal covering of the smallest size is called the covering number of the graph.

Theorem 5.4:

A covering g of a graph is minimal if and only if g contains no paths of length three or more.

Proof:

Part 1: If g has no paths of length ≥ 3, then g is minimal.

Suppose a covering g contains no paths of length three or more. These means all connected components of the graph formed by the edges in g must be paths of length $0, 1$ or 2 (which are stars with 1 or 2 edges).

If we removed any edge from a path of length 1 or 2 at least one vertex previously incident to that edge will no longer be incident to any edge in the remaining set.

Therefore, no edge can be removed without violating the covering property, making g a minimal covering.

Part 2: If g is minimal, the g has no paths of length ≥ 3.

We can prove this contradiction, Suppose g is a minimal covering but contains a path P of length 3 or more denoted by vertices and edges as $v_1 - e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - \cdots$.

Consider an edge e_2 in the middle of this path incident to v_2 and v_3. If we remove e_2 the vertex v_2 is still covered by e_1 and v_3 is still covered by e_3.

Since all other vertices in the graph covered by edges in g other than e_2, removing e_2 still leaves all vertices covered. This is contradicts the assumption that g is minimal (ie That no edge can be removed).

Therefore, a minimal covering cannot contain a path of length 3 or more.

Hence proved.

FOUR-COLOR PROBLEM:

So far we have considered proper coloring of vertices and proper coloring of edges. Let us briefly consider the proper coloring of regions in a planar graph (embedded on a plane or sphere). Just as in coloring of vertices and edges, the regions of a planar graph are said to be properly colored if no two contiguous or adjacent regions have the same color. (Two regions are said to be adjacent if they have a common edge between them. Note that one or more vertices in common does not make two regions adjacent.) The proper coloring of regions is also called map coloring, referring to the fact that in an atlas different countries are colored such that countries with common boundaries are shown in different colors.

Once again we are not interested in just properly coloring the regions of a given graph. We are interested in a coloring that uses the minimum number of colors. This leads us to the most famous conjecture in graph theory. The conjecture is that every map (i.e., a planar graph) can be properly colored with four colors. The four-color conjecture, already referred to in Chapter 1, has been worked on by many famous mathematicians for the past 100 years. No one has yet been able to either prove the theorem or come up with a map (in a plane) that requires more than four colors.

That at least four colors are necessary to properly color a graph is immediate from Fig:5.5, and that five colors will suffice for any planar graph will be shown shortly.

Two remarks may be made here in passing. Paradoxically, for surfaces more complicated than the plane (or sphere) corresponding theorems have been proved. For example, it has been proved that seven colors are necessary and sufficient for properly coloring maps on the surface of a torus. Second, it has been proved that all maps containing less than 40 regions can be properly colored with four colors. Therefore, if in general the four-color conjecture is false, the counterexample has to be a very complicated and large one.

Vertex Coloring Versus Region Coloring: From Unit-V we know that a graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G^*, and vice versa. Thus the four-color conjecture can be restated as follows: Every planar graph has a chromatic number of four or less.

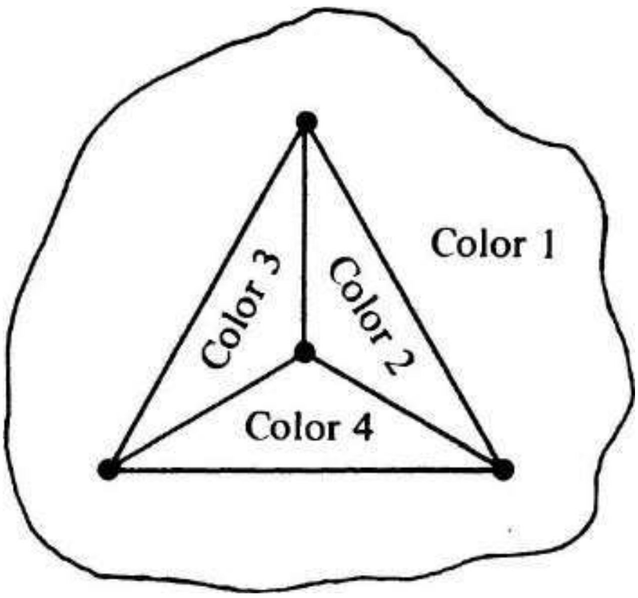


Fig: 5.5

Five-Color Theorem: We shall now show that every planar map can be properly colored with five colors:

Theorem 5.5:

The vertices of every planar graph can be properly colored with five colors.

Proof:

For a planar graph G with $n \leq 5$ vertices, trivially assign a unique color to each vertex. The theorem is holds.

Assume all planar graphs with n vertices and 5 colorable. Let G be a graph with $n + 1$ vertices. According, to Euler's formula, every planar graph ha a vertex v with degree $d(v) \leq 5$. Remove v from G to form $G' = G - v$.

By the induction hypothesis, G' is 5-colourable.

Case 1: $d(v) < 5$ or neighbours use < 5 colors. If v has fewer than 5 neighbours, or two neighbour share the same color, there is at least one color available for v among the 5 colors.

Case 2: $d(v) = 5$ and all neighbour use 5 colors. Let the neighbours be v_1, v_2, v_3, v_4, v_5 in clockwise order colored c_1, c_2, c_3, c_4, c_5 respectively.

Consider the subgraph $H_{1,3}$ consisting of vertices colored c_1 and c_3. If v_1 and v_3 are in diffferent connected components of $H_{1,3}$, we can switch the colors 1 and 3 in the component containing v_1. Now v_1 is color 3, and color 1 is free for v.

If v_1 and v_3 are connected, there is a path of alternating colors 1 and 3 between them. Because the graph is planar this path together with v seperates v_2 from v_4.

Therefore, there cannot be a path of alternating colors 1 and 3 between v_2 and v_4. We can safely swap colors 2 and 4 in the component containing v_2 allowing v to take color 2.

Thus, the 5-coloring is valid for $n + 1$ vertices.

Hence proved.

DIRECTED GRAPHS:

The graphs studied in this book so far have been undirected graphs. No direction was assigned to the edges in a graph. An edge e_k between vertices v_i and v_j could be considered as going from vertex v_i to vertex v_j or from v_j to v_i. In this unit we shall consider directed graphs–graphs in which edges have directions.

Many physical situations require directed graphs. The street map of a city with one-way streets, flow networks with valves in the pipes, and electrical networks, for example, are represented by directed graphs. Directed graphs are employed in abstract representations of computer programs, where the vertices stand for the program instructions and the edges specify the execution sequence. The directed graph is an invaluable tool in the study of sequential machines. Directed graphs in the form of signal-flow graphs are used for system analysis in control theory.

Most of the concepts and terminology of undirected graphs are also applicable to directed graphs. For example, the planarity of a graph does not depend on whether the graph is directed or undirected, and therefore unit - v is applicable to both directed and undirected graphs. The same is true for most other topics covered so far. It would be wasteful to devote another eight chapters to the study of directed graphs, mostly repeating, with minor changes, what has already been said. In this chapter, therefore, we shall mainly bring out those properties of directed graphs that are not shared by undirected graphs.

Definition 5.5: (Directed Graph)

A directed graph (or a digraph for short) G consists of a set of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2, \ldots\}$, and a mapping Ψ that maps every edge onto some ordered pair of vertices (v_i, v_j). As in the case of undirected graphs, a vertex is represented by a point and an edge by a line segment between v_i and v_j with an arrow directed from v_i to v_j.

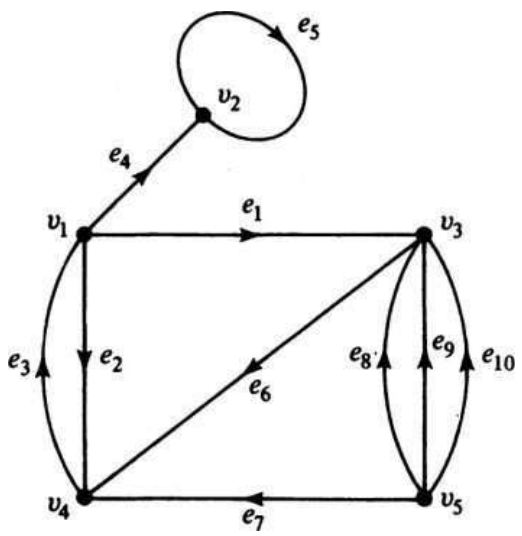


Fig: 5.6

In Fig: 5.6 shows a digraph with five vertices and ten edges. A digraph is also referred to as an oriented graph.

In a digraph an edge is not only incident on a vertex, but is also **incident out** of a vertex and **incident into** a vertex. The vertex v_i, which edge e_k is incident out of, is called the **initial vertex** of e_k. The vertex v_j, which e_k is incident into, is called the **terminal vertex** of e_k. In Fig: 5.6, v_5 is the initial vertex and v_4 is the terminal vertex of edge e_7. An edge for which the initial and terminal vertices are the same forms a **self-loop**, such as e_5. (Some authors reserve the term arc for an oriented or directed edge. We use the term edge to mean either an undirected or a directed edge. Whenever there is a possibility of confusion, we shall explicitly state directed or undirected edge).

The number of edges incident out of a vertex v_i is called the **out-degree** (**or out-valence or outward demidegree**) of v_i and is written $d^+(v_i)$. The number of edges incident into v_i is called the in-degree (or invalence or inward demidegree) of v_1 and is written as $d^-(v_i)$.

In Fig:5.6, $d^+(v_1) = 3, d^-(v_1) = 1$

$$d^+(v_2) = 1, d^-(v_2) = 2$$

$$d^+(v_5) = 4, d^-(v_5) = 0$$

It is not difficult to prove that in any digraph G the sum of all in-degrees is equal to the sum of all out-degrees, each sum being equal to the number of edges in G. that is,

$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i)$$

An **isolated vertex** is a vertex in which the in-degree and the out-degree are both equal to zero. A vertex v in a digraph is called pendant if it is of degree one, that is, if $$d^+(v) + d^-(v) = 1.$$

Two directed edges are said to be parallel if they are mapped onto the same ordered pair of vertices. That is, in addition to being parallel in the sense of undirected edges, parallel directed edges must also agree in the direction of their arrows. In Fig: 5.6, edges e_8, e_9 and e_{10} are parallel, whereas edges e_2 and e_3 are not.

Since many properties of directed graphs are the same as those of undirected ones, it is often convenient to disregard the orientations of edges in a digraph. Such an undirected graph obtained from a directed graph G will be called the undirected graph corresponding to G.

On the other hand, given an undirected graph H, we can assign each edge of H some arbitrary direction. The resulting digraph, designated by $\overline{H}$ is called an orientation of H (or a digraph associated with H). Note that while a given digraph has a unique (within isomorphism) undirected graph corresponding to it, a given undirected graph may have "different" orientations possible. This is why we say the undirected graph corresponding to a digraph, but an orientation of a graph.

This brings us to the question: When are two digraphs considered to be the same or isomorphic?

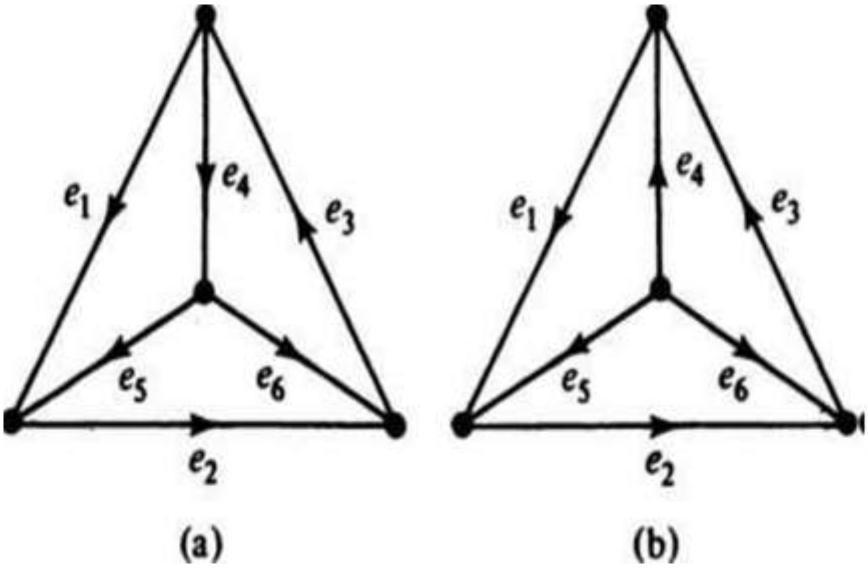


Fig: 5.7

Isomorphic Digraphs:

Isomorphic graphs were defined such that they have identical behavior in terms of graph properties. In other words, if their labels are removed, two isomorphic graphs are indistinguishable. For two digraphs to be isomorphic not only must their corresponding undirected graphs be isomorphic, but the directions of the corresponding edges must also agree. For example, Fig: 5.7 shows two digraphs that are not isomorphic, although they are orientations of the same undirected graph.

SOME TYPES OF DIGRAPHS:

Like their undirected sisters, digraphs come in many varieties. In fact, due to the choice of assigning a direction to each edge, directed graphs have more varieties than undirected ones. **Simple Digraphs:** A digraph that has no self-loop or parallel edges is called a simple digraph (Figs: 5.7 and 5.8, for example). Asymmetric Digraphs: Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or antisymmetric.

Symmetric Digraphs: Digraphs in which for every edge (a, b) (i.e., from vertex a to b) there is also an edge (b, a). A digraph that is both simple and symmetric is called a simple symmetric digraph. Similarly, a digraph that is both simple and asymmetric is simple asymmetric.

Complete Digraphs: A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge. For digraphs we have two types of complete graphs. A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex (Fig. 9-3), and **a complete asymmetric digraph** is an asymmetric digraph in which there is exactly one edge between every pair of vertices Fig: 5.7. A complete asymmetric digraph of n vertices contains $\frac{n(n-l)}{2}$ edges, but a complete symmetric digraph of n vertices contains $n(n-1)$ edges. A complete asymmetric digraph is also called a **tournamen**t or a **complete tournament**. A digraph is said to be balanced if for every vertex vi the in-degree equals the out-degree. That is, $d^{+}(v_i) = d^{-}(v_i)$. (A balanced digraph is also referred to as a **pseudosymmetric** digraph, or an isograph). A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

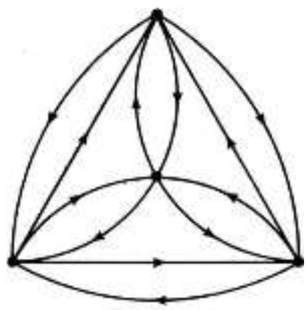


Fig: 5.8

DIRECTED PATHS AND CONNECTEDNESS:

Walk, Path and Circuit in a directed graph, in addition to being what they are in the corresponding undirected graph, have the ordered consideration of orientation. For example in Fig: 5.6, the sequence of vertices and edges $v_5 e_8 v_3\ e_6 v_4 e_3 v_1$ is a path directed from v_5 to v_1 whereas $v_5 e_7 v_4 e_6 v_3 e_1 v_1$(although a path in the corresponding undirected graph) has no such consistent direction from v_5 to v_1. A distinction must be made between these two types of paths. It is natural to call the first one a directed path from v_5 to v_1 and the second one a semi path. The word path in a digraph could mean either a directed graph or semi path and similarly for walk,circuit and cutsets. More precisely:

A **directed walk** from a vertex v_i to v_j is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j, such that each edge is oriented from the vertex preceding it to the vertex following it. Of course, no edge in a directed walk appears more than once, but a vertex may appear more than once, just as in the case of undirected graphs. A **semi-walk** in a directed graph is a walk in the corresponding undirected graph, but is not a directed walk. A **walk** in a digraph can mean either a directed walk or a **semi-walk**.

The definitions of **circuit, semi-circuit**, and **directed circuit** can be written similarly. Let us turn to Fig: 5.6 once more. The set of edges $\{e_1, e_6, e_3\}$ is a directed circuit. But $\{e_1, e_6, e_2\}$ is a semi-circuit. Both of them are circuits.

Connected Digraphs: In a graph (i.e., undirected graph) was defined as connected if there was at least one path between every pair of vertices. In a digraph there are two different types of paths. Consequently, we have two different types of connectedness in digraphs. A digraph G is said to be **strongly**

connected if there is at least one directed path from every vertex to every other vertex. A digraph G is said to be **weakly connected** if its corresponding undirected graph is connected but G is not strongly connected. In Fig: 5.7 one of the digraphs is strongly connected, and the other one is weakly connected. The statement that a digraph G is connected simply means that its corresponding undirected graph is connected and thus G may be strongly or weakly connected. A directed graph that is not connected is dubbed as disconnected.

Since there are two types of connectedness in a digraph, we can define two types of components also. Each maximal connected (weakly or strongly) subgraph of a digraph G will still be called a component of G. But within each **component** of G the maximal strongly connected subgraphs will be called the **fragments** (or **strongly connected fragments**) of G.

The digraph in Fig: 5.9 consists of two components. The component g_1 contains three **fragments** $\{e_1, e_2\}, \{e_5, e_6, e_7, e_5\}$ and $\{e_{10}\}$. Observe that e_3, e_4 and e_9 do not appear in any fragment of g_1.

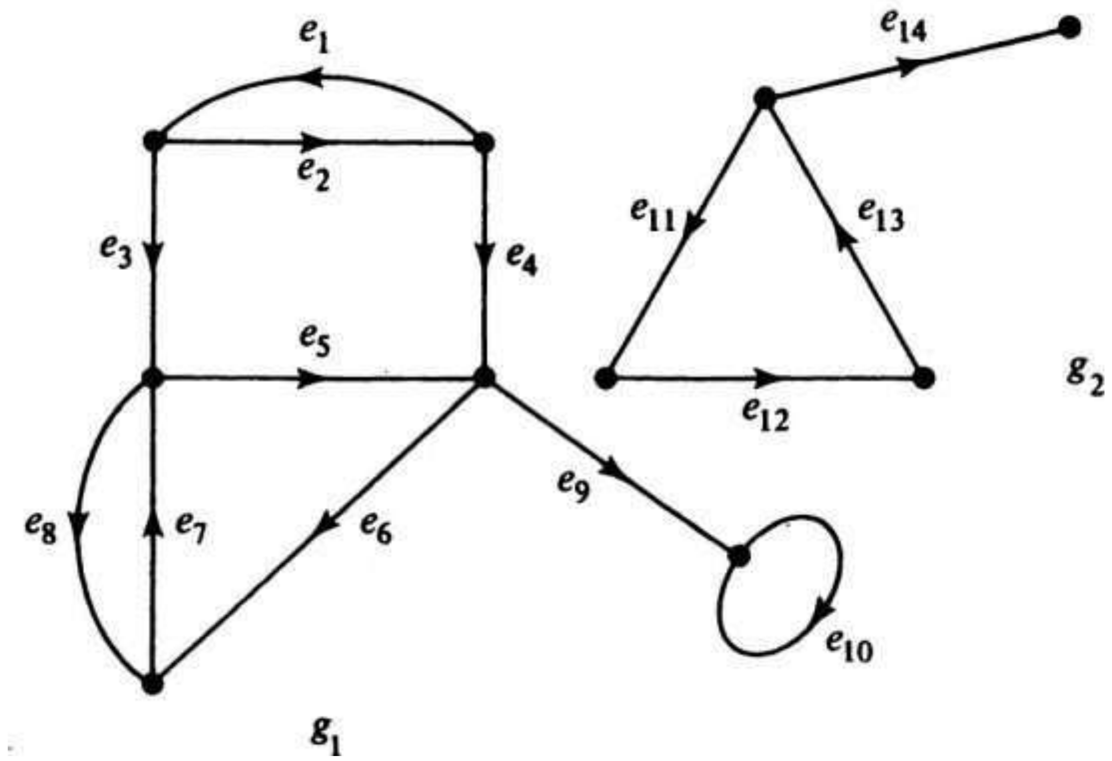


Fig: 5.9

Condensation: The condensation G_c of a digraph G is a digraph in which each strongly connected fragment is replaced by a vertex and all directed edges from one strongly connected component to another are replaced by a single directed edge. The condensation of the digraph G in Fig: 5.9 is shown in Fig:5.10.

Two observations can be made from the definition:

1. The condensation of a strongly connected digraph is simply a vertex.

2. The condensation of a digraph has no directed circuit.

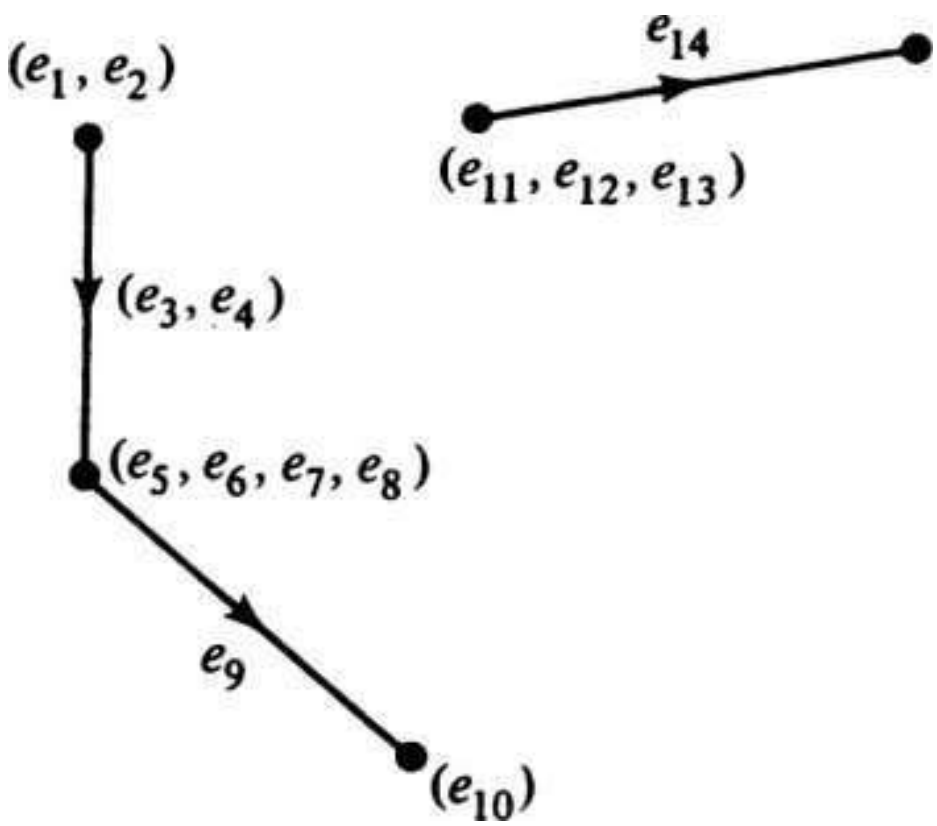


Fig: 5.10

Accessibility: In a digraph a vertex b is said to be accessible (or reachable) from vertex a if there is a directed path from a to b. Clearly, a digraph G is strongly connected if and only if every vertex in G is accessible from every other vertex.

EULER DIGRAPHS:

The notion of the Euler graph can be extended to digraphs also. In a digraph G a closed directed walk (i.e., a directed walk that starts and ends at the same vertex) which traverses every edge of G exactly once is called a directed Euler line. A digraph containing a directed Euler line is called an Euler digraph. The graph in Fig: 5.11 is an Euler digraph, in which the walk $a\ b\ c\ d\ e\ f$ is an Euler line.

When is a digraph an Euler digraph? Clearly, the digraph must be connected, with the possible exception of isolated vertices; otherwise, every edge cannot be traversed in one walk. In fact, an Euler digraph must be strongly connected, although every strongly connected digraph need not be an Euler digraph, Theorem 5.6 provides a very simple test for determining whether or not a digraph has an Euler line.

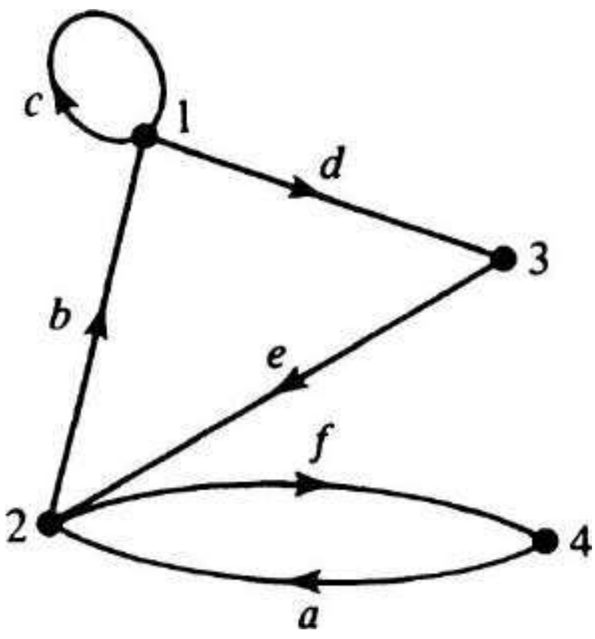


Fig: 5.11

Theorem 5.6:

A digraph G is an Euler digraph if and only if G is connected and is balanced [i.e., $d^-(v) = d^+(v)$ for every vertex v in G].

Proof:

Necessary part:

Assume G is an Euler graph contaiing an Euler circuit C.

By defintion, an Euler circuit C includes every edge of G. If G contains no isolated vertices, the existence of a single circuit that visits every edge implies that every vertex must be reachable from every other vertex via edges in the circuit. Thus, the graph is (weakly) connected.

Let v be any vertex in G. Each time the Euler circuit C passes through v, it must enter v via one edge and leave v via another. Entering v contributes 1 to the in-degree $d^{-}(v)$ and leaving v contributes 1 to the out-degree $d^{+}(v)$. Because the circuit is closed and uses every edge exactly once, the total number of times we enter v must equal the total number of times we leave it. Therefore, $d^{-}(v) = d^{+}(v)$ for every vertex v.

Sufficient part:

Assume G is connected and balanced $d^{-}(v) = d^{+}(v)$ for all v. We can contruct an Euler circuit using an iterative process.

Strat at any vertex v_0. Since, $d^{+}(v) = d^{-}(v) > 0$ for all vertices with edges, we can always leave any vertex we enter. By following unused edges, we must eventually return to v_0, forming a directed cycle C_1.

If C_1 includes all edges of G we are done. If not, because G is connected, there must be a vertex w on C_1 that is incident to edges not in C_1. Since, the original graph was balanced and C_1 itself is balanced, the remaining graph $G' = G - E(C_1)$ is also balanced. Strating from w in G', we can find another cycle C_2. We then splice C_2 into C_1 at vertex w.

Repeate this process untill all edges are included. Since, the graph is finite, we will eventually form an large circuit that contains every edge exactly once.

Hence proved.

Teleprinter's Problem: How long is a longest circular (or cyclic) sequence of l's and 0's such that no subsequence of r bits appears more than once in the sequence? Construct one such longest sequence.

Solution:

Since there are 2^r distinct r-tuples formed from 0 and 1, the sequence can be no longer than 2^r bits long. Using Theorem 5.6, we shall construct a circular sequence 2^r bits long with the required property that no subsequence of r bits be repeated.

Construct a digraph G whose vertices are all $(r - 1)$-tuples of 0's and l's. Clearly, there are 2^{r-1} vertices in G. Let a typical vertex be $\alpha_1\, \alpha_2 \dots \alpha_{r-1}$, where $\alpha_i = 0$ or 1.

Draw an edge directed from this vertex $(\alpha_1\, \alpha_2 \dots \alpha_{r-1})$ to each of two vertices $(\alpha_2\, \alpha_3 \dots \alpha_{r-1}.0)$ and $(\alpha_2\, \alpha_3 \dots \alpha_{r-1}.1)$, label these directed edges $(\alpha_2\, \alpha_3 \dots \alpha_{r-1}.0)$ and $(\alpha_2\, \alpha_3 \dots \alpha_{r-1}.1)$ respectively. Draw two such edges directed from each of the 2^{r-1} vertices. (A self-loop will result in each of the two cases when $\alpha_1 = \alpha_2 = \cdots = \alpha_{r-1} = 0$ or 1.)

The resulting digraph is an Euler digraph because for each vertex the in-degree equals the out-degree (each being equal to two). A directed Euler line in G consists of the 2^r edges, each with a distinct r-bit label. The labels of any two consecutive edges in the Euler line are of the form $\alpha_1\alpha_2 \dots \alpha_{r-1}\, \alpha_r;\ \alpha_2\, \alpha_3 \dots \alpha_r\, \alpha_{r+1}$. That is, the $r - 1$ trailing bits of the first edge are identical to the $r - 1$ leading bits of the second edge. Thus in the sequence of 2^r bits, made of the first bit of each of the edges in the Euler line, every possible subsequence of r bits occurs as the label of an edge; and since no two edges have the same label, no subsequence occurs more than once. The circular arrangement is achieved by joining the two ends of the sequence.

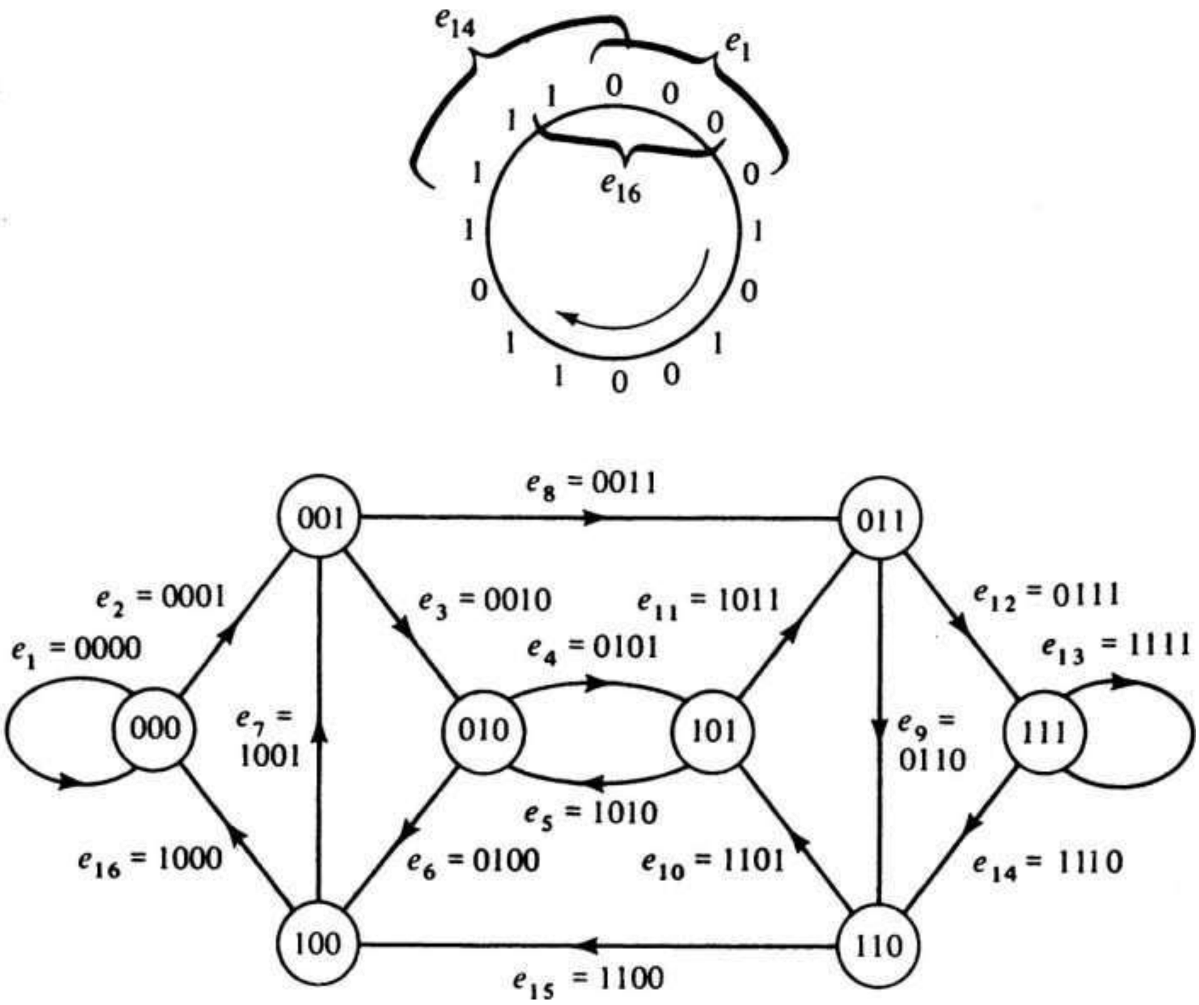


Fig: 5.12

For $r = 4$, the graph in Fig:5.12 illustrates the procedure of obtaining such a maximum-length sequence. One such sequence is 0000101001101111 corresponding to the walk $e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7\ e_8\ e_9\ e_{10}\ e_{11}\ e_{12}\ e_{13}\ e_{14}\ e_{15}\ e_{16}$.

This problem is also encountered in locating the position of a rotating drum with the surface carrying two different types of marks. The problem of the rotating drum is similar to that of a feedback shift register. The quest for a word in which each arrangement of r letters of a given alphabet appears exactly once, encountered in cryptography, is also the same problem.

It is not difficult to see that the alphabet size need not be 2. It could be any number m. In that case, the maximum-length sequence is m^r symbols long. The in-degree and out-degree of each vertex in the corresponding Euler digraph equals m, rather than two.

Number of Euler Lines: In Fig: 5.12 there is more than one Euler line. In fact, the digraph has 16 distinct Euler lines. (Note that rotations of the same sequence of edges are not considered distinct.) Finding the number of distinct directed Euler lines in a given Euler digraph is also of interest in many applications. This problem of enumeration was solved by N. G. deBruijn in 1946 for those regular Euler digraphs in which the in-degree and out-degree of every vertex were exactly two, the digraph in Fig: 5.12.

The number of distinct Euler lines in a blalanced, connected digraph G can be obtained by counting certain types of spanning trees in G.